# Plane acoustic-gravity waves

## Acoustic-gravity waves

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# Outline

# propagation of plane waves

## Warning about phase velocity concept in 2D

- In a 1D space, the phase velocity of a wave is a clear concept : velocity of iso-phase points (e.g. crests). in 1D, the phase velocity is a scalar.
- In a multidimensional space (2D hereafter), the phase velocity of a plane wave is a more ambiguous concept. Two different concepts are useful (see explanations below) :

  - "absolute phase velocity"
  - "phase velocity along a direction **d**".

# Propagation of plane waves (in 2D)

$$\psi = \psi_0 \exp i(kx + mz + \omega t) = \psi_0 \exp i(\mathbf{K} \cdot \mathbf{r} - \omega t)$$

## Absolute phase velocity in 2D

- velocity of phase lines (i.e. along the direction of propagation) : $c = \frac{\omega}{|\mathbf{K}|}$
- this is a scalar (a vector $\mathbf{c} = \frac{\omega}{|\mathbf{K}|^2}\mathbf{K}$ can be constructed, but it does not offer any useful practical use).

## Phase velocity along a direction **d**

- velocity of iso-phase points along the straight lines of direction **d**.
- this is a scalar $c_{\mathbf{d}} = \frac{\omega}{\mathbf{K} \cdot \mathbf{d}}$.
- this is NOT the projection of **c** along **d**.

## A more efficient concept : phase slowness vector

- scalar slowness = inverse of scalar speed
- vector slowness = slowness of phase lines along the direction of propagation : $\mathbf{s} = \mathbf{K}/\omega$.
- The phase slowness along a direction **d** is now the projection of the phase slowness vector along **d**
- The phase speed along a direction **d** is the inverse of the phase slowness along **d**.

# Frequency for plane waves :

**general dispersion equation :**

$$\gamma\omega^4 - c^2\left[\gamma k^2 + \nu^2 + 1/4H^2\right] + k^2 N^2 c^2 = 0$$

Hydrostatic :

$$\omega^2 = \frac{k^2 N^2}{\nu^2 + 1/4H^2}$$

nonhydrostatic :

$$\omega^2 = \frac{1}{2}\left[c^2\left(k^2 + \nu^2 + 1/4H^2\right) \pm \sqrt{c^2\left(k^2 + \nu^2 + 1/4H^2\right)^2 - 4k^2 N^2 c^2}\right]$$

# Frequency for gravity plane waves :

Hydrostatic :

$$\omega^2 = \frac{k^2 N^2}{\nu^2 + 1/4H^2}$$

nonhydrostatic :

$$\omega^2 = \frac{1}{2}\left[c^2\left(k^2 + \nu^2 + 1/4H^2\right) - \sqrt{c^2\left(k^2 + \nu^2 + 1/4H^2\right)^2 - 4k^2 N^2 c^2}\right]$$

since usually $4N^2 \ll 1$, for nonhydrostatic gravity waves we have :

$$\omega^2 \approx \frac{k^2 N^2}{k^2 + \nu^2 + 1/4H^2}$$

# Propagation of gravity waves in NH

## phase velocities along $x, \sigma$

$$
\begin{aligned}
c_x &= \frac{\omega}{k} \approx \frac{N}{\sqrt{k^2+\nu^2+1/4H^2}} \\
c_\sigma &= \frac{\omega}{\nu} \approx \frac{N}{\sqrt{k^2+\nu^2+1/4H^2}} \left(\frac{k}{\nu}\right)
\end{aligned}
$$

## group velocity vector

$$
\begin{aligned}
\mathbf{V_g}\big|_x &= \frac{\partial\omega}{\partial k} \approx \frac{N}{\sqrt{k^2+\nu^2+1/4H^2}^3} \left(\nu^2+1/4H^2\right) \\
\mathbf{V_g}\big|_\sigma &= \frac{\partial\omega}{\partial\nu} \approx \frac{N}{\sqrt{k^2+\nu^2+1/4H^2}^3} \left(-k\nu\right)
\end{aligned}
$$

# Propagation of gravity waves in NH Boussinesq case

## phase velocities along $x, \sigma$

$$c_x \approx \frac{N}{\sqrt{k^2+\nu^2}} \qquad c_\sigma \approx \frac{N}{\sqrt{k^2+\nu^2}}\left(\frac{k}{\nu}\right)$$

## group velocity

$$\mathbf{V_g}\big|_x \approx \frac{N}{\sqrt{k^2+\nu^2}^3}\left(\nu^2\right) \qquad \mathbf{V_g}\big|_\sigma \approx \frac{N}{\sqrt{k^2+\nu^2}^3}\left(-k\nu\right)$$

## wave geometry

- $\mathbf{V_g}$ always perpendicular to the wave vector $\mathbf{K}$
- vertical component of group velocity always opposite sign to the one of phase velocity

# Propagation of gravity waves in Hydrostatic Boussinesq case

## phase velocities along $x, \sigma$

$$c_x \approx \frac{N}{\nu} \qquad c_\sigma \approx \frac{Nk}{\nu^2}$$

## group velocity

$$\mathbf{V_g}\big|_x \approx \frac{N}{\nu} \qquad \mathbf{V_g}\big|_\sigma \approx -\frac{Nk}{\nu^2}$$

## wave geometry

- $\mathbf{V_g}$ always perpendicular to the wave vector $\mathbf{K}$
- vertical component of group velocity always opposite sign to the one of phase velocity

# Comparisonof orographic gravity waves in Boussinesq case

## orographic waves :

- basic wind $U$
- stationary waves $\Rightarrow \quad c_x = -U$

## Hydrostatic Boussinesq :

$$c_x + U = 0 \quad \Rightarrow \quad \mathbf{V_g}\big|_x + U = 0$$

The propagation of energy is vertical

## nonhydrostatic Boussinesq :

$$c_x + U = 0 \quad \Rightarrow \quad \mathbf{V_g}\big|_x + U = \frac{Uk^2}{k^2 + \nu^2}$$

The propagation of energy is slantwise (leeward)