Design of the hybrid vertical coordinate η
(case of a domain with $\pi_{\rm top} = 0$)

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file : memoeta0.tex

1 Introduction

In the litterature about η hybrid coordinate, the functions A and B are often said to be "arbitrary" functions. However, even if A and B are chosen such as:

$$A(0) = 0 \quad (1)$$
$$A(1) = 0 \quad (2)$$
$$B(0) = 0 \quad (3)$$
$$B(1) = 1 \quad (4)$$

and even if B is taken positive and monotonously growing and A positive, this does not guarantee that this choice results in a coordinate (i.e. that $\pi(\eta) = A(\eta) + B(\eta)\pi_s$ is monotonous in η). It should be noticed here that as soon A is not identically zero, there always exists a surface-pressure value below which a given choice of (A, B) ceases to be a coordinate. This value is given by:

$$\pi_s = \max_{\eta\in[0,1]} \left[\frac{-dA/d\eta}{dB/d\eta} \right] \quad (5)$$

The coordinate property is then violated in the area where $dA/d\eta$ is negative and large, i.e. in the area where the "terrain-followingness" exhibits its fastest decrease. However, for applications to meteorology, we know that π_s cannot be smaller than an absolute given limit, as e.g. $\pi_s|_{\rm Everest}$, hence we are allowed to choose coordinates for which A is not identically zero, that is, coordinates which actually are hybrid coordinates, provided that (5) cannot be satisfied anywhere in the domain.
Currently, in the ARPEGE and Aladin models, this constraint is set by aborting if A and B are such that the following condition is violated:

$$\max \left[\frac{-dA/d\eta}{dB/d\eta} \right] > \pi_{\rm sMin} \qquad \text{[condition for arborting]} \quad (6)$$

where $\pi_{\rm sMin}$ is currently set to 450 hPa. This condition is currently checked through a test in `FACADI.F90`, that is, very late, in the forecast model itself.

Note: Since this value 450 hPa is larger than the effective $\pi_s|_{\rm Everest}$, this constraint should be made more stringent in the future, when the resolution will increase so as the effective height of the Mount Everest will pass below 450 hPa.
It should also be noted that a set of function (A, B) satisfying the positivity constraint too closely with the actual minimum surface pressure of the model would probably result in an inappropriate coordinate, since the layer depth above the maximum orography would virtually vanish. Hence in any practical application, a "safety margin" must be taken, in order that the depth of layers remain non-vanishing. This can be obtained by setting $\pi_{\rm sMin}$ significantly below the minimum observed value in the application.

Previously, the specification of A and B was made by using an external program, in which some basic parameters were specified as an input, and (A, B) were specified *directly*. However, this program did not allow all the flexibility that could be desired for such a tool, and more annoying, it did not prevent outputs of (A, B) which were not coordinates for the model in practice. The constraint (6) was ignored by this program, and even if it was diagnosed, the program did not indicate how to modify an unproper choice of (A, B) in order to obtain an appropriate coordinate.
The aim of this paper is to present the alternative way chosen to define the functions A and B, allowing more flexibility than with the previous method, and allowing an intuitive way to correct an ill-defined initial coordinate.

The chosen approach is to define separately the "stretching" and the "hybridicity" profiles of the coordinate, to start with, and only then, to combine these profiles in order to get the sought A and B functions. The guidelines are presented in section 2, the definition of the stretching function in the section 3 and the definition of the hybridicity function in section 4. Practical informations are provided in sections 5 and 6.
A further modification for the definition of A and B functions has been implemented in 2006. This modification, intended to allow some local refinement of the resolution near the tropopause is documented at the end of the document, in section 7, before conclusion.

N.B.: This paper is restricted to the most usual case in NWP of a "vertically unbounded" atmosphere, that is, when the pressure at the top of the domain vanishes. Another paper (file: memoetatop.tex), more general but less simple, deals with the case of a "vertically bounded" atmosphere for which the top pressure is not identically zero. Bounded domains are not used in operational NWP so far, but are useful for academic studies or inter-comparison exercises, where the extension of the vertical domain is prescribed.

2 Guideline

Three basic parameters are first set: the number of "half levels" for the output $L+1$ (where L is the number of layers or "full levels"), a reference pressure $\pi_{00} = 101325.$ Pa, and the minimum surface pressure π_{sMin} for which the set A, B must define a valid coordinate.
In order to practically define the namelists, an level index $\tilde{l}$ is introduced for "half levels". By definition we have:

$$\tilde{l} \in \{0, 1, ..., L\}\,. \tag{7}$$

In the ARPGE/IFS terminology, the index $\tilde{l}$ describes the index for interfaces between layers. The index $\tilde{l} = 0$ represents the top of the domain, while $\tilde{l} = L$ represents the bottom of the domain.

However, the A and B functions will be defined through continuous functions. Therefore a new independant variable has to be defined. A real independant variable x is introduced, with

$$x \in [0, 1]. \tag{8}$$

The value $x = 0$ represents the top of the domain, while $x = 1$ represents the bottom of the domain. Once continuous versions of A and B will have been defined, we will finally discretise them through:

$$x(\tilde{l}) = \tilde{l}/L \tag{9}$$

The constraint (6) is a given fact that we have to take into account as soon as in the definition of A, B functions. The principle of the present approach to fullfil this requirement is to define:

- A "stretching" function : this function defines how the resolution of the η coordinate should vary along the vertical when $\pi_s = \pi_{00}$. Let $m(x)$ be this function, referred to as the "stretching function" hereafter. This stretching function is thus defined by:

$$m : [0, 1] \longrightarrow [0, 1] \tag{10}$$
$$x \longrightarrow y = m(x) \tag{11}$$

- An "hybridicity" function: this function defines the "proportion" of terrain-following (σ) and pure-pressure (π) coordinates that we want to have at a given level y in the stretched grid. Let $h(y)$ be thus function, referred to as the "hybridicity function" hereafter. This hybridicity function is thus defined by:

$$h : [0, 1] \longrightarrow [0, 1] \tag{12}$$
$$y \longrightarrow z = h(y) \tag{13}$$

Finally the A and B functions are defined directly from the stretching and the hybridicity functions:

$$A(x) = \pi_{00}\{m(x) - h[m(x)]\} \tag{14}$$
$$B(x) = h[m(x)] \tag{15}$$

For a given surface pressure π_s, the derivative of the pressure with respect to the coordinate writes, for a given level x_0:

$$\frac{1}{\pi_{00}}\left(\frac{d\pi}{dx}\right)_{(x=x_0)} = \left(\frac{dm}{dx}\right)_{(x=x_0)}\left\{1 - \frac{(\pi_{00}-\pi_s)}{\pi_{00}}\left(\frac{dh}{dy}\right)_{(y=m(x_0))}\right\} \tag{16}$$

For this choice to define a coordinate under the constraint (6), it is therefore sufficient that h is chosen in such a way that:

$$\frac{dh}{dy} < X \equiv \frac{\pi_{00}}{(\pi_{00} - \pi_{\mathrm{sMin}})} \tag{17}$$

This defines a second constraint, that is easy to handle practically. The current value of X is 1.8 (from the test in `FACADI`).

The slope of the m function decribes the local vertical resolution in pressure. The value of m at the surface is chosen equal to $m(1) = 1$. At the top, for operational namelists, we currently choose to impose $m(0) = 0$, thus implicitly constraining the pressure to be zero at the top of the domain.
The "actual" hybridicity itself would be defined by $B/(B + A/\pi_{00})$ and thus given by $h(y)/y$, i.e. the slope of the straight-line joining the origin to the point $(y, h(y))$ on the curve of h (hence, $h(y)$ is *not* exactly the hybridicity itself but the hybridicity multiplied by y)· Since A and B are positive, we must have $h(y) \leq y$ for any value of y. The proposed definition of $h(y)$ makes the expression of the constraint (17) more simple than what it would be with the exact hybridicity $h(y)/y$. In order to fullfil $\pi = \pi_s$ at the surface, we must impose $h(1) = 1$.

Except the constraint (17), the choice of m and h functions is now free. In order to restrict this hudge amount of degrees of freedom, we choose an approach in which some caracteristic points for these functions are explicitely specified. Then the functions are defined in such a way that they pass exactly by theses points with suitable continuity and derivability properties. For these specifications, the atmosphere is supposed to be discretized with L layers.

3 Definition of the stretching function

For the definition of the stretching function m, the vertical domain (consisting in L layer) is conventionally divided into 5 parts:
- the bottom model layer
- the "PBL"
- the "troposphere" (which is indeed the central part of the domain)
- the "stratosphere"
- the uppermost model layer

The four interfaces between these subdomains are then used as characteristic points, each of them being specified in practice through a level index l, and a corresponding pressure π (for a surface pressure $\pi_s = \pi_{00}$). Thus we define the following characteristic points:
- bottom of uppermost model layer: $x_1 = 1/L$, $y_1 = \delta\pi_1/\pi_{00}$
- bottom of the "stratosphere": $x_2 = (N_{\mathrm{Strato}})/L$, $y_2 = (\pi_{\mathrm{Strato}})/\pi_{00}$
- top of the "PBL": $x_3 = (L - N_{\mathrm{PBL}})/L$, $y_3 = (\pi_{\mathrm{PBL}})/\pi_{00}$
- top of bottom model layer: $x_4 = (L - 1)/L$, $y_4 = (\pi_{00} - \delta\pi_L)/\pi_{00}$

The physical meaning of all above parameters is defined in section 5.

N.B.: Please note that if the specification of the uppermost characteristic point is to be specified from a desired value of π_1 at the first full level, then the knowledge of the value of LAPRXPK is required in order to know the corresponding pressure-depth of the first layer $\delta\pi_1$.

The stretching function is then defined sub-domain by sub-domain:

uppermost sub-domain (for $0 < x \leq x_1$):

$m(x) = (y1/x1)\ x$ (used only if an analytic expression of A, B is needed)

"Strato" sub-domain: (for $x_1 < x \leq x_2$):

We note $d_1 = \frac{x_1 y_2 - x_2 y_1}{x_1}\ (x_2 - x_1)^{-\alpha_1}$

$m(x) = x\ (y1/x1) + d_1\ (x - x_1)^{\alpha_1}$

where α_1 is an arbitrary exponent (typically $\alpha_1 \in [1,5]$).

We note: $y_2' = dm/dx$ at $x = x_2$

"PBL" sub-domain: (for $x_3 < x \leq x_4$):

We note $d_3 = \frac{(1 - x_4)(1 - y_3) - (1 - x_3)(1 - y_4)}{(1 - x_4)}\ (x_4 - x_3)^{-\alpha_3}$

$m(x) = 1 - [(1 - y_4)/(1 - x_4)]\ (1 - x) - d_3\ (x_4 - x)^{\alpha_3}$

We note: $y_3' = dm/dx$ at $x = x_3$

where α_3 is an arbitrary exponent (typically $\alpha_3 \in [1,5]$).

bottom sub-domain: (for $x_4 < x \leq 1$):

$m(x) = 1 - [(1 - y_4)/(1 - x_4)]\ (1 - x)$ (used only if an analytic expression of A, B is needed)

"Tropo" sub-domain: (for $x_2 < x \leq x_3$):

We note: $\Delta x = (x_3 - x_2)$, $\Delta y = (y_3 - y_2)$, $s = \Delta y/\Delta x$.

$m(x) = y_2 + (x - x_2)\ y_2' + (x - x_2)^2\left[\Delta x(s - y_2') + (x - x_3)(y_2' + y_3' - 2s)\right]/\Delta x^2$

This is nothing else than the differentiable fit between the characteristic points (x_2, y_2) and (x_3, y_3).
The characteristic points of the stretching function are depicted in Fig. 1: the two extreme parts of the curve are straight lines. The derivative of the curve is continuous at every characteristic point.

4 Definition of the hybridicity function

The hybridicity function h is build by using two characteristic points and checks the fulfilment of the constraint represented by (17). The vertical domain of the stretched coordinate "y" is divided into three sub-domains:

- pure terrain-following bottom-subdomain: may never be empty, must contain at least the surface point $y = 1$, may contain the whole domain $y \in [0,1]$.

- pure pressure top-subdomain : must at least contain the top level $y = 0$.

- hybrid intermediate-subdomain : must contain exactly the space left by the two previous layers, should be empty only if the whole domain is in pure terrain-following coordinate.

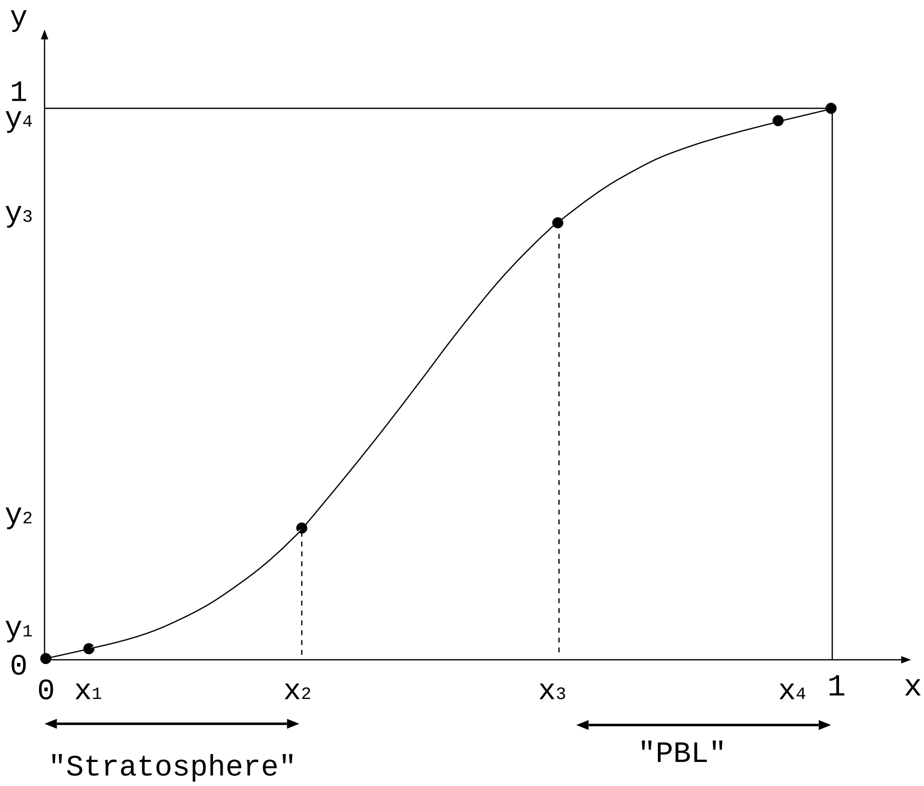

Figure 1: *Characteristic points of the stretching function* $y = m(x)$

The two interfaces between these sub-domains are used as characteristic points which are defined by their stretched coordinate y, computed in practice from a level index:

- stretched level of the top of pure terrain-following layer: $y_\sigma = m[(L - N_\sigma)/L]$
- stretched level of the bottom of pure pressure layer: $y_\pi = m[(N_\pi)/L]$

where N_σ is the number of pure terrain-following levels *in addition to the mandatory surface one*, and N_π is the number of pure σ and π levels respectively.
The number of pure pressure levels N_π is free in $\{0, \ldots, L\}$, however choosing a too large value for N_π will lead to a violation of (17) for quite large values of $\pi_{\rm sMin}$, and therefore is not recommanded.
The number of the non-surface terrain-following levels is also free in $\{0, \ldots, L\}$,

$$N_\sigma \in \{0, ..., L\} \tag{18}$$

Choosing $N_\sigma = L$ and $N_\pi = 0$ means that the whole domain is described with a pure terrain-following coordinate, whilst $N_\sigma = 0$ means that the only pure terrain-following level is the surface one. Specifying $N_\pi = 0$ and $N_\sigma < L$ means that the only pure pressure level is the top one and an hybrid coordinate is used.
The constraint (17) imposes:

$$y_\pi \leq y_\sigma \frac{X - 1}{X} \tag{19}$$

where X is defined in (17).
For the hybridicity function, we first define:

$$d_1 = \alpha_h \frac{y_\sigma^2}{(y_\sigma - y_\pi)} \tag{20}$$

$$d_2 = 1 + \alpha_h \frac{(y_\sigma)}{(y_\sigma - y_\pi)} \tag{21}$$

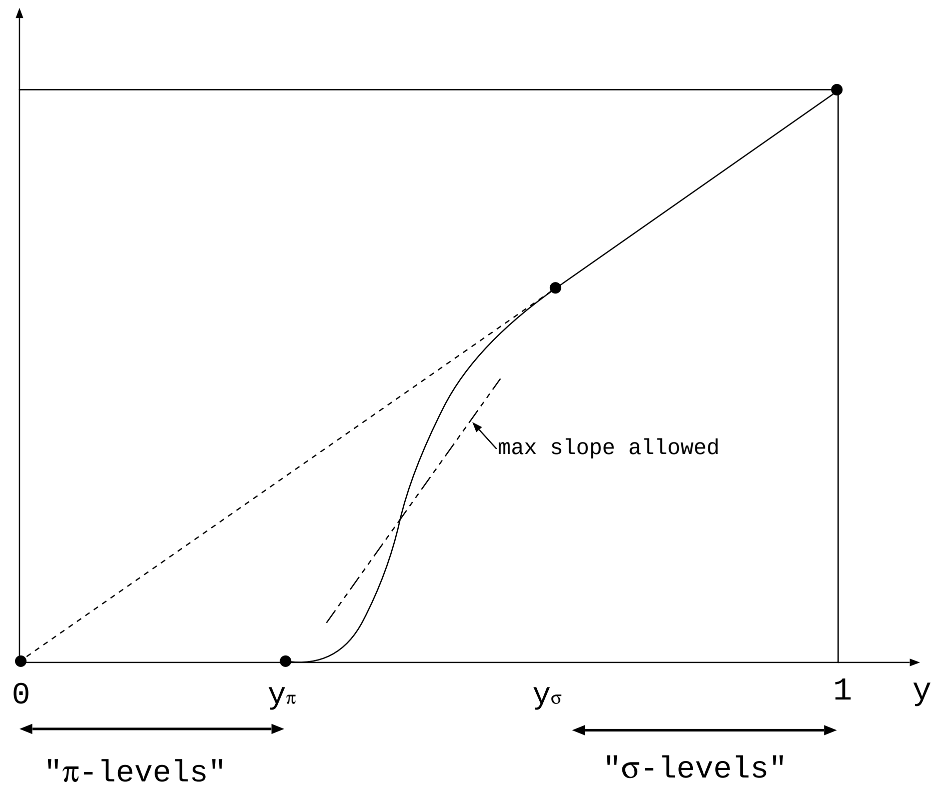

Figure 2: *Characteristic points of the hybridicity function $z = h(y)$. In this example, the coordinate definition is inappropriate and should be revised.*

where α_h is an arbitrary exponent (with typically $\alpha_h \in [-1, -3]$). The value $\alpha_h = -1$ results in a linear increase of h with y Then the hybridicity function h is defined by:

$$h(y) = 0 \quad \text{for } 0 \leq y \leq y_\pi.$$

$$h(y) = \frac{d_1}{d_2 - [(y - y_\pi)/(y_\sigma - y_\pi)]^{\alpha_h}} \quad \text{for } y_\pi < y < y_\sigma.$$

$$h(y) = y \quad \text{for } y_\sigma \leq y \leq 1.$$

It is seen that a pure σ coordinate is obtained by setting $N_\sigma = L$ and $N_\pi = 0$. When using actually hybrid coordinates in ARPEGE, it is a tradition to set $N_\sigma \geq 1$ and $N_\pi \geq 1$, i.e. there is at least two pure-pressure levels at top and at least two pure terrain-following levels at the bottom.
The characteristic points of the hybridicity function are depicted in Fig. 2. The derivative of the curve is continuous at every characteristic point (except if $\alpha_h = -1$ for which the intermediate curve is a straight line joining the characteristic points). In the exemple shown in the figure, the coordinate is not a valid one since the slope in the hybrid part is larger than the maximum allowed slope. In this case, it would be necessary to decrease y_π and/or increase y_σ.

5 Free parameters

The free parameters for defining a vertical coordinate are then:

Physical parameters:
- L : total number of levels,
- N_{PBL} : number of levels in the "PBL",
- π_{PBL} : pressure of the "PBL" top,
- N_{Strato} : number of levels in the "stratosphere",
- π_{strato} : pressure of the "tropopause",

- $\delta\pi_L$: pressure depth of the bottom model layer,
- π_1 : pressure depth of the top model layer.
- N_σ : number of pure terrain following levels in addition to the mandatory surface one,
- N_π : number of pure pressure levels.

Mathematical parameters:
- α_1 : exponent for the "stratosphere",
- α_3 : exponent for the "PBL",
- α_h : exponent for the hybridicity.

6 Tuning of mathematical parameters

The mathematical parameters α_1, α_3, α_h have no immediate meaning. It is enough to know that they define the rate at which the curves in the sub-domain depart from linear functions joining the characteristic points. For instance, $\alpha_1 = \alpha_3 = 1$ means linear functions, while e.g. $\alpha_1 = \alpha_3 = 5$ means high resolutions very concentrated near the edges of the domain. Similarly, $\alpha_h = -1$ means an hybridicity linearly increasing with the level index, and $\alpha_h = -3$ means a smoother variation. The impact of α_h is however quite weak.
Since the setting of the mathematical parameters α_1, α_3 is not fully intuitive, the program provides reasonable default values:

$$\alpha_1 = \left(\frac{0.8\, y_3 - y_2}{x_3 - x_2} - \frac{y_1(x_2 - x_1)}{x_1 y_2 - x_2 y_1} \right) \tag{22}$$

and:

$$\alpha_3 = \left(\frac{1.4\, y_2 - y_3}{x_2 - x_3} - \frac{(1 - y_4)(x_4 - x_3)}{(1 - x_4)(1 - y_3) - (1 - x_3)(1 - y_4)} \right) \tag{23}$$

For the parameter α_h, as outlined above, the impact is weak and values slightly smaller than -1 should always be a good choice.
In any case, it is recommended to plot the obtained coordinate and related functions (m, h, depth of layers...) in order to check that the result is "aesthetically convenient".

7 Modification to allow more resolution near the tropopause

A specific requirement for NWP applications is to have the capability of locally increasing the resolution near the tropopause. For this a modification of the design has been implemented in 2006. The approach which was chosen does not use "control points" as previously, but modifies the function $m(x)$ in the interval $[x_2, x_3]$ in order to decrease the value of m between these two values as seen in Fig. 3.
The original stretching function $m(x)$ is replaced by a modified stretching function $[m(x).f(x)]$ inside the interval $[x_2, x_3]$. Outside this interval, the original stretching function is kept. The function f is a polynomial which leaves values of m untouched at both edges, thus:

$$f(x_2) = f(x_3) = 1 \tag{24}$$

Moreover, f leaves untouched as much successive derivatives as possible at both edges, in order to insure smoothness of the solution. Polynomial of degrees 4, 5 and 6 were implemented. For a degree 6 polynomial, the expression is as follows:

$$f(x) = 1 - a \left(\frac{2}{x_3 - x_2} \right)^6 (x - x_2)^3 (x_3 - x)^3 , \qquad \text{for } x \in [x_2, x_3], \tag{25}$$

therefore insuring two successive derivatives untouched at both edges. The tuning parameters for this new functionality are the degree of the polynomial f and the positive parameter a (only the sixth degree is documented here). The minimum value of f, at $x_m = (x_3 + x_2)/2$ is $f(x_m) = (1 - a)$.
If $a = 0$ the original stretching function is kept, whilst if $a > 0$ the stretching is modified as in Fig. 3. This figure is made with a 55-levels domain. It is seen that the resolution is refined near 24th the level, in areas (indicated by a cross) where the pressure is about 170 hPa. The value $a = 0.486$ was chosen in this example.

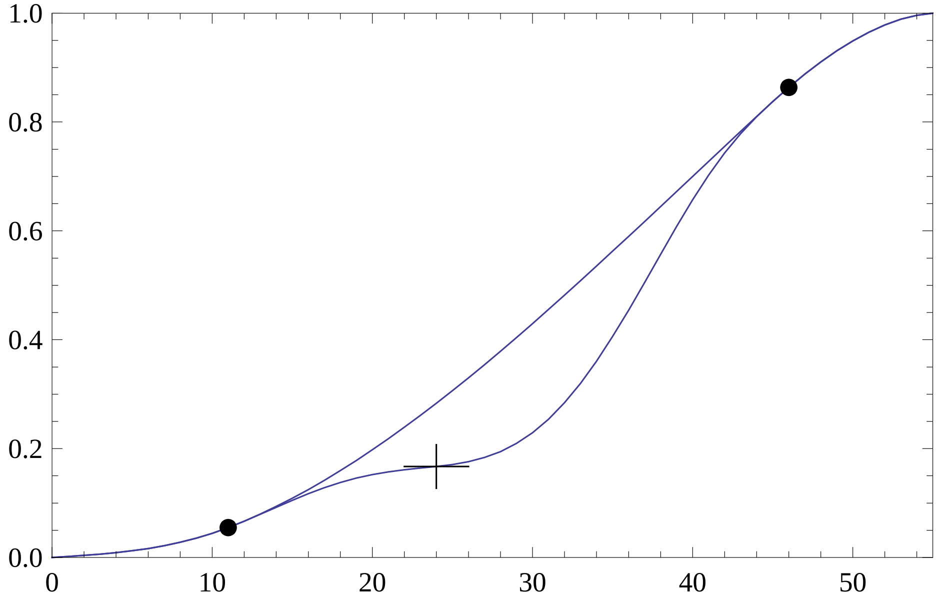

Figure 3: *Modification of the stretching function $m = m(x)$ by a function $f(x)$ between the two characteristic points x_2 and x_3, for a given 55-levels discretization. Abcissa : level index ; ordinate value of m. Upper curve : unmodified function, lower curve : modified function with a sixth-order polynomial as in text (with $a = 0.486$). The cross indicates the centre of the area of increased resolution, the two dots indicate x_2 (left) and x_3 (right) characteristic points.*

Care must be taken of course not to have a negative slope for the stretching function, which might result of a too large value of a. For NWP applications, the area of refined resolution is required to be around the physical (actual) tropopause, therefore, the characteristic point x_2 needs of course to be placed *above the actual tropopause* since the effect of f on resolution refinement is felt rather far below x_2. In the example shown on the figure, the actual tropopause would be near level 24, and the characteristic point x_2 needed to be set to the level 11. When this new functionality is used, the characteristic point x_2 should no longer be termed with the misleading name "tropopause level".

8 Conclusions

A method for specifying as rationally as possible the A and B functions for the need of NWP application has been described. The method allows a flexible specification of resolution near surface and top of the domain while insuring a proper definition of the coordinate above a specified maximum orographic elevation (i.e. a specified minimum surface-pressure)
The modification brought in the last section allows a local refinement of the resolution near the tropopause, for specific needs of some recent NWP applications. This refinement uses an additional degree of freedom (tuning parameter).