D.2. Available recursive digital filters in ALADIN

Discrete formulations of the filtered state for a recursive filter of order K

$$\overline{x}_n = \sum_{k=0}^{K} a_k x_{n-k} + \sum_{k=1}^{K} b_k \overline{x_{n-k}} \quad , \ n \geq K$$

Using lower order filters for $n < K$

or

$$\overline{x}_n = \sum_{p=0}^{n} c_p x_{n-p} \ , \ 0 \leq n \leq N$$

(⇒ also called "Infinite Impulse Response" filters)

The time-span must be at least $2K\Delta t$ $(N \geq .2K)$.

Ideal (as N → +∞) response function

$$H(\omega) = \frac{\sum_{k=0}^{K} a_k e^{-ik\omega\Delta t}}{1 - \sum_{k=1}^{K} b_k e^{-ik\omega\Delta t}} = \mathcal{H}(i\frac{\omega}{\omega_c})$$

where $\mathcal{H}$ is the transfer function of the equivalent analog filter of order K, defined by :

$1/\mathcal{H}$ is a polynomial function of order K, $\mathcal{H}(0) = 1$ $|\mathcal{H}(\mathrm{i})| = 1/\sqrt{2}$

$$\mathcal{H}(\frac{1}{\mu_c} \cdot \frac{z-1}{z+1}) = \frac{\sum_{k=0}^{K} a_k z^{-k}}{1 - \sum_{k=1}^{K} b_k z^{-k}} \ , \qquad \mu_c = tg(½\omega_c\Delta t) = tg(\frac{\pi\Delta t}{T_c})$$

Effective response

$$H_n(\omega) = \sum_{p=0}^{n} c_p e^{-ip\omega\Delta t}$$

The response of recursive filters is <u>complex</u>.

Examples

"QUICK-START" filter of order K (optimizing the time-span)

$$\mathcal{H}(s) = \left(\frac{\sigma_K}{s+\sigma_K}\right)^K, \quad \sigma_K = \sqrt{1/\left(\sqrt[K]{2}-1\right)}$$

$$\mathcal{H}(s) = \left(\frac{\sigma_2}{s+\sigma_2}\right)^2, \quad \sigma_2 = \sqrt{\sqrt{2}+1}, \text{ for } K=2$$

"FLAT-DELAY" filter of order K (optimizing the phase response)

$$\mathcal{H}(s) = \frac{\mathcal{B}_K(0)}{\mathcal{B}_K(s\Omega_K)}, \quad \mathcal{B}_K \text{ Bessel polynomial of order } K, \quad \left|\mathcal{B}_K(i\Omega_K)\right|^2 = 2\left|\mathcal{B}_K(0)\right|^2$$

$$\mathcal{H}(s) = \frac{\gamma}{s^2+s\sqrt{3\gamma}+\gamma}, \quad \gamma = \left(1+\sqrt{5}\right)/2, \text{ for } K=2$$

"BUTTERWORTH" filter of order K (optimizing the amplitude response)

$$\mathcal{H}(s) = 1/\prod_{k=0}^{K-1}(s-e^{i\theta_k}), \quad \theta_k = \frac{\pi}{2}\cdot(1+\frac{1}{K})+\frac{k\pi}{K}$$

$$\mathcal{H}(s) = \frac{1}{s^2+s\sqrt{2}+1}, \text{ for } K=2$$