The role of tunings in fractional time-stepping schemes for the non-linear diffusion equation

Piet Termonia, Geert Smet and Joris Van den Bergh

Royal Meteorological Institute of Belgium

INTRODUCTION

We address the role of the tuning of the parameters of subgrid parameterisations of diffusive processes in numerical models. The stability and accuracy of a set of fractional time-splitted integration schemes for the non-linear diffusion equation is examined. The best value of the time-decentering parameter γ is established.

We investigate what the potential is for improving the scheme by a "tuning" of the exchange coefficients of the diffusion equation and what the effect is of using a parameterisation in one model that has been tuned in another model.

We suggest that a sequential time splitting of the diffusive processes with respect to their forcing is more attractive than the parallell one. Secondly, it is shown that the stability criterium for γ differs from the one derived by Kalnay and Kanamitsu for a concurrent scheme: smaller values can ensure stability.

ANALYTICAL RESULTS

Method of Kalnay and Kanamitsu (1988) stability analysis, applied to fractional time-stepping schemes. Study of Dubal et al (2004), concerning correct representation of steady state, extended to non-linear case.

Forced non-linear damping equation:

$$\frac{\partial X}{\partial t} = D(X) + S = -(KX^P)X + S\,, \qquad (1)$$

Discretisation done with a two time-level scheme, treating physics terms S, D(X) with fractional time-stepping. Forcing is symmetrized around damping, and assumed constant in this section.

• Parallel scheme:

$$\begin{aligned} X^1 - X &= D(X^1, X)\Delta t \\ X^2 - X &= S\Delta t \\ X^+ - X &= \sum_i (X^i - X)\,. \end{aligned}$$

• Sequential scheme:

$$\begin{aligned} X^* &= X + \eta S\Delta t \\ X^{**} &= X^* + D(X^{**}, X^*)\Delta t \\ X^+ &= X^{**} + (1-\eta)S\Delta t\,. \end{aligned}$$

• Discretisation (explicit coefficient, extrapolated temperature, 2TL scheme):

$$D(X^+, X) = -KX^P\left[\gamma X^+ + (1-\gamma)X\right]\,.$$

The decentering parameter γ determines the amount of implicitness of the numerical scheme.

Best decentering for reproducing stationary state:

• Linear case (Dubal et al, 2004): $\gamma = \eta\,.$

• Non-linear case (P=1): $\gamma = \eta + \dfrac{\eta}{1+\eta\Delta t\sqrt{KS}}\,. \qquad (2)$

• For general P: the stationary state cannot be reproduced if γ is outside the range [η, (P+1)η]. Specifically, the parallel scheme cannot reproduce it. This makes the sequential scheme more attractive.

Stability analysis:

• Reminder: Kalanay-Kanamitsu criterium for concurrent scheme:

$$\gamma > \frac{\alpha - 1}{\alpha} \qquad \text{with} \qquad \alpha \equiv KX_{ss}\Delta t\,,$$

with X_{ss} the stationary solution.

• Stability criterium for sequential (and parallel) scheme (P=1):

$$\gamma > \frac{\alpha - 1 + \eta\beta}{\alpha + \beta(\eta + \frac{1}{2})} \qquad \text{with} \qquad \beta \equiv KS\Delta t^2\,.$$

• Stability analysis for general P:

– for the purpose of stability, it does not matter how the forcing is split around the diffusion.

– less decentering is needed to assure stability than would be in the case of a concurrent treatment of forcing and damping. *Examples: figure 1, figure 2.*

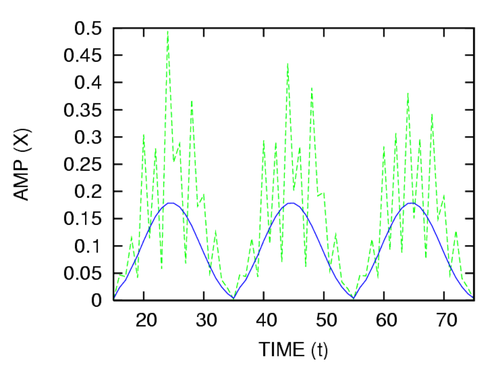

Figure 1: Sequential (blue continuous line) vs. concurrent scheme (green dashed) for small decentering parameter (0.58).

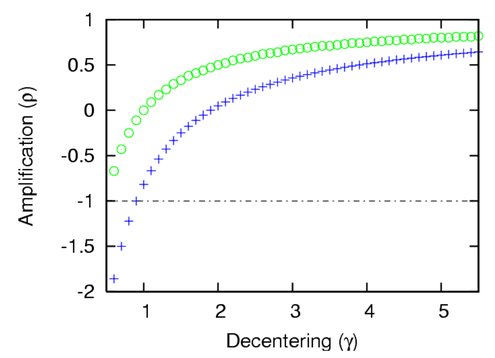

Figure 2: Amplification factor as function of the decentering parameter for sequential scheme (green circles) with η = 1 and concurrent scheme (blue crosses). The stable region is above the horizontal line (amplification factor -1).

NUMERICAL EXPERIMENTS

We consider equation (1) with a periodic forcing

$$S(t) = 1 + \sin\left(2\pi\frac{t}{20}\right)\,,$$

as in Kalnay and Kanamitsu (1988), for the cases P = 1, 2, 3, 4 and K=10,1000. To create an accurate reference solution for each case, we use the concurrent scheme with γ = 0.5 and a very small timestep Δt = 0.0001. We tune the γ of the sequential scheme using equation (2) with S=1, see figure 3.

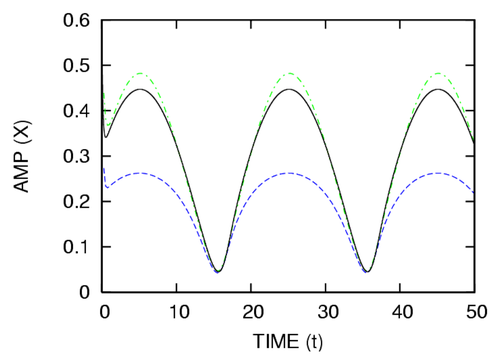

Figure 3: Tuned versus untuned for the sequential scheme (with K=10, P=1, η=1 and Δt = 1/4). The black continuous line is the reference solution, the blue dashed line is the sequential scheme with untuned γ=1, and the green dash-dotted line is the sequential scheme with tuned γ=1.56.

TUNING OF EXCHANGE COEFFICIENT K

We try to obtain the correct stationary state by tuning the exchange coefficient K, instead of tuning the decentering parameter γ. Replace physical K with tuned value K_t:

$$K_t = \frac{S}{(X_{ss} + S\eta)^P (X_{ss} + (\eta - \gamma)S\Delta t)}\,, \qquad (3)$$

where X_{ss} is the stationary state $X_{ss} = (S/K)^{1/(P+1)}$, using the physical K.

Numerical example given in figure 4: physical K given by K_{phys} = 10. Using equation (3) with γ=η=1 gives a value for the tuned K given by K_{tuned} = 2.4. Tuning γ gives a more accurate result (γ=1.24).

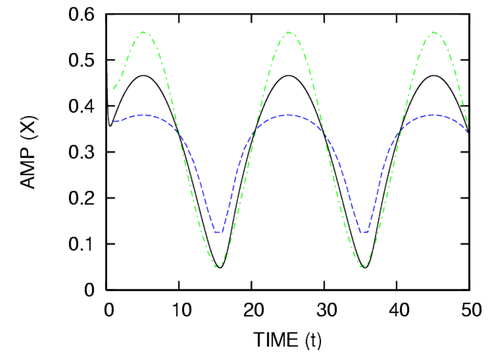

Figure 4: Solution with K tuned vs reference solution and solution with γ tuned (for P=1, K=10, S=1, Δt=1, η=1). Black continuous line: reference, blue dashed line: K tuned and γ=η=1, green dash-dotted line: γ tuned.

In figure 5, we tune the exchange coefficient K for the parallel scheme (η=0) to obtain K^{para}. We then plug K^{para} into the sequential scheme (η=1), while leaving the forcing S and time step Δt unchanged. This mimics what is currently being tried in full NWP models. For instance, within the ALADIN and HIRLAM consortia it has been decided to exchange physics parameterisations developed in different models to be plugged in a dynamical core that is commonly shared.

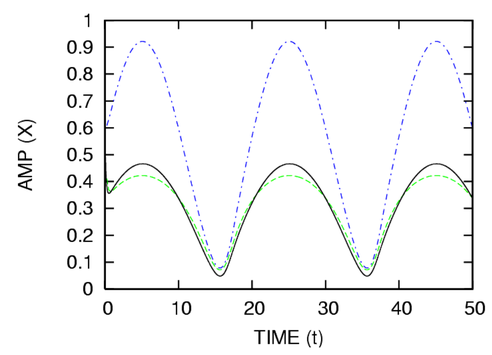

Figure 5: Sequential scheme with P=1, K=10, S=1, Δt=1/4, η=1. Black continuous line: reference, dashed line: K tuned, dash-dotted line: wrong tuning of K for parallel scheme (η=0).

CONCLUSIONS

• Smaller values can be used for the decentering parameter when using a **sequential time-stepping scheme** as opposed to a concurrent scheme. In numerical experiments, we found a threshold **γ > 0.8** (independent of η) to suffice for **stability** purposes, for range P=0-4, K=10-1000. This does not mean that a sequential scheme is more accurate. But it should be kept in mind when choosing γ in more realistic situations.

• **Accuracy** can be greatly reduced in a sequential-split scheme at large time steps, due to **incorrect representation of the stationary state**. We give an expression for **optimising γ** to overcome this.

• Other method: **tuning of exchange coefficient K**: toy model for the tuning of parameters in physics parameterisation packages.

• This improves representation of the stationary state, but depends sensitively on P and K_{phys}. Also on η, i.e. the type of scheme used.

• Example: tune K for parallel scheme (η=0) and plug into sequential scheme (η=1) → large difference, loss of accuracy.

REFERENCES

• Dubal M., Wood N., Staniforth A. 2004. *Analysis of parallel versus sequential splittings for time-stepping physical parameterizations*. Mon. Weather Rev. **132**, 121-132.

• Kalnay E., Kanamitsu K. 1988. *Time Schemes for Strongly Nonlinear Damping Equations*. Mon. Wea. Rev. **116**, 1945-1958.

20th ALADIN Workshop & HIRLAM ASM 2010, 13-16 April, 2010, Cracow, Poland.