The "Horizontally Explicit and Vertically Implicit" time scheme applied on the fully compressible Euler's equations in mass-based

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$$[\partial_t + \mathbf{V}\cdot\nabla + \dot{\eta}\partial_\eta]\,\mathbf{V} + RT\left(\frac{\nabla\pi}{\pi} + \nabla q\right) + \nabla\phi + \mathrm{Y} = 0, \quad \text{(hori. mot.)}$$

$$[\partial_t + \mathbf{V}\cdot\nabla + \dot{\eta}\partial_\eta]\,w + \frac{g}{m}\partial_\eta\left[\pi(\mathrm{e}^q - 1)\right] = 0, \quad \text{(vert. mot.)}$$

$$[\partial_t + \mathbf{V}\cdot\nabla + \dot{\eta}\partial_\eta]\,T + \frac{RT}{C_v}\left(\nabla\cdot\mathbf{V} - \frac{\mathrm{e}^q}{H}\frac{\pi}{m}\partial_\eta w + \mathrm{X}\right) = 0, \quad \text{(energy)}$$

$$[\partial_t + \mathbf{V}\cdot\nabla + \dot{\eta}\partial_\eta]\,q + \frac{\dot{\pi}}{\pi} + \frac{C_p}{C_v}\left(\nabla\cdot\mathbf{V} - \frac{\mathrm{e}^q}{H}\frac{\pi}{m}\partial_\eta w + \mathrm{X}\right) = 0, \quad \text{(presure)}$$

$$\partial_t q_s + \pi_s\int_0^1 \nabla\cdot(m\mathbf{V})\mathrm{d}\eta = 0. \quad \text{(continuity)}$$

The fully compressible system (*EE*)

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Horizontal adjustment terms : *horizontal propagation of fast waves*

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Vertical adjustment terms : *vertical propagation of fast waves*

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Horizontal adjustment terms	:	*horizontal propagation of fast waves*
Vertical adjustment terms	:	*vertical propagation of fast waves*
Advection terms	:	*wind transport*

A efficient algorithm

General constraints for schemes

Explicit schemes are stable only if the *CFL*-criterion is verify (Courant *et al.* (1928)) :

$$\max\left(\frac{U\Delta t}{\Delta x};\frac{w\Delta t}{\Delta z};\frac{c_s\Delta t}{\Delta x};\frac{c_s\Delta t}{\Delta z}\right)\leq \mathrm{Cte}.$$

Additional constraints from the NWP :

(i) continue : high velocity of fastest phenomena ($c_s \gg U$)

(ii) discrete : high vertical resolution ($\Delta z \ll \Delta x$)

To overcome constraints :

- *(i)* : filtered systems (Lipps & Hemler [1982], Duran [1989], Dubos & Voitus [2014]) $\Longrightarrow$ solving a Poisson elliptic problem (non-local).
- *(i)* & *(ii)* : implicit schemes (Robert *et al.* [1972], Bénard [2010]) $\Longrightarrow$ solving a Helmholtz elliptic problem (non-local).

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The *HEVI*-schemes alternative

Properties currently checked by NWP models

- More local algorithms $\Longleftrightarrow$ schemes more explicit.
- The most restrictive CFL-criterion is that relating to the Courant number of the vertical propagation of acoustic waves.
- Each column is processed by a single processor.

Properties of more effective schemes in this context

- Explicit scheme for the most communicative direction.
  - $\rightarrow$ Explicit treatment of the terms responsible for the horizontal propagation of acoustic waves.
  - $\rightarrow$ stability is defined by the CFL-criterion $c_s \Delta t / \Delta x \leq \mathrm{Cte.}$
- Implicit scheme for the direction generating the largest CFL-criterion.
  - $\rightarrow$ Implicit treatment of the terms responsible for the vertical propagation of acoustic waves.
  - $\rightarrow$ Relax of the discrete constraint *(ii)*.

Temporal discretization of (EE) with a double Butcher's tableau

$$\partial_t X = \mathscr{M}(X) = \underbrace{\mathscr{E}(X)}_{\text{"explicit"}} + \overbrace{\mathscr{I}(X)}^{\text{"implicit"}} .$$

Runge-Kutta scheme for each part :

$$\frac{X^{(j)} - X^t}{\Delta t} = \sum_{k=1}^{j-1} \tilde{a}_{jk} \mathscr{E}(X^{(k)}) + \sum_{k=1}^{j} a_{jk} \mathscr{I}(X^{(k)}),$$

$$\frac{X^{t+\Delta t} - X^t}{\Delta t} = \sum_{j=1}^{\nu} \tilde{b}_j \mathscr{E}(X^{(j)}) + \sum_{j=1}^{\nu} b_j \mathscr{I}(X^{(j)}).$$

Coefficients are resumed in a double Butcher's tableau $\{\tilde{c}, \tilde{\mathcal{A}}, \tilde{b}\}$ and $\{c, \mathcal{A}, b\}$ (with $\tilde{\mathcal{A}} = (\tilde{a}_{ji})_{1 \le j, i \le \nu}$ et $\mathcal{A} = (a_{ji})_{1 \le j, i \le \nu}$) and must verify Pareschi & Russo [2005] conditions to be a second order schemes.

Best candidates analyzed by Lock *et al.* [2014]

UJ3(1,3,2) from Ullrich & Jablonowski [2012] :

0	0					
0	0	0				
1	0	1	0			
1/2	0	1/4	1/4	0		
1	0	1/6	1/6	2/3	0	
1	0	1/6	1/6	2/3	0	0
	0	1/6	1/6	2/3	0	0

0	0					
1/2	1/2	0				
1/2	1/2	0	0			
1/2	1/2	0	0	0		
1/2	1/2	0	0	0	0	
1	1/2	0	0	0	0	1/2
	1/2	0	0	0	0	1/2

ARK2(2,3,2) from Giraldo [2012] :

0	0		
2γ	2γ	0	
1	$1-\alpha$	α	0
	δ	δ	γ

0	0		
2γ	γ	γ	
1	δ	δ	γ
	δ	δ	γ

où $\alpha = 1/2 + \sqrt{2}/3$, $\delta = \sqrt{2}/4$ et $\gamma = 1 - \sqrt{2}/2$

Trap2(2,3,2)(-1) from Weller *et al.* [2013] :

0	0			
0	0	0		
1	1/2	1/2	0	
1	1/2	0	1/2	0
	1/2	0	1/2	0

0	0			
1	1	0		
1	1/2	0	1/2	
1	1/2	0	0	1/2
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0	0					
1/2	1/2	0				
1/2	1/2	0	0			
1/2	1/2	0	0	0		
1/2	1/2	0	0	0	0	
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1/2	1/2	0	0			
1/2	1/2	0	0	0		
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	1/2	0	0	1/2

The use of RK-IMEX HEVI scheme

Application on a 2D-linear system without orography

The linear system is obtained by a linearisation of the fully Euler's equations around a $\overline{X}$ state which is in hydrostatic balance, uniform, isothermal ($\overline{T}$), with a uniform wind $\overline{U}$ and without orography.

$$\partial_t u + \overline{U}\nabla u + R\mathcal{G}\nabla T + R\overline{T}\left(\mathcal{I} - \mathcal{G}\right)\nabla q + R\overline{T}\nabla q_s = 0,$$

$$\partial_t w + \overline{U}\nabla w - g(\tilde{\partial} + \mathcal{I})q = 0,$$

$$\partial_t T + \overline{U}\nabla T + \frac{R\overline{T}}{C_v}\left(\nabla u - \frac{1}{\overline{H}}\tilde{\partial}w\right) = 0,$$

$$\partial_t q + \overline{U}\nabla q - \mathcal{S}\nabla u + \frac{C_p}{C_v}\left(\nabla u - \frac{1}{\overline{H}}\tilde{\partial}w\right) = 0,$$

$$\partial_t q_s + \mathcal{N}\nabla u = 0,$$

where $\mathscr{G}$, $\mathscr{S}$ and $\mathscr{N}$ are vertical integrate operators and $\tilde{\partial}$ and $(\tilde{\partial} + \mathcal{I})$ vertical derivative operators, and $\overline{H} = R\overline{T}/g$.

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$$\partial_t w + \overline{U}\nabla w - g(\tilde{\partial}+\mathcal{I})q = 0,$$

$$\partial_t T + \overline{U}\nabla T + \frac{R\overline{T}}{C_v}\left(\nabla u - \frac{1}{\overline{H}}\tilde{\partial}w\right) = 0,$$

$$\partial_t q + \overline{U}\nabla q - \mathcal{S}\nabla u + \frac{C_p}{C_v}\left(\nabla u - \frac{1}{\overline{H}}\tilde{\partial}w\right) = 0,$$

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$$\partial_t w + \overline{U}\nabla w - g(\tilde{\partial} + \mathcal{I})q = 0,$$

$$\partial_t T + \overline{U}\nabla T + \frac{R\overline{T}}{C_v}\left(\nabla u - \frac{1}{\overline{\overline{H}}}\tilde{\partial} w\right) = 0,$$

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UFpreF : *U-Forward and pressure-Forward*

Weller *et al.* [2013]

Application on a 2D-linear system without orography

$$u^{(j)} = u^t - \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk} \left[\overline{U} \nabla u^{(k)} + R\mathcal{G} \nabla T^{(k)} + R\overline{T} \left(\mathcal{I} - \mathcal{G} \right) \nabla q^{(k)} + R\overline{T} \nabla q_s^{(k)} \right]$$

$$w^{(j)} - a_{jj} \Delta t g (\tilde{\partial} + \mathcal{I}) q^{(j)} = w^t - \Delta t \sum_{k=1}^{j-1} \left(\tilde{a}_{jk} \overline{U} \nabla w^{(k)} - a_{jk} g (\tilde{\partial} + \mathcal{I}) q^{(k)} \right)$$

$$T^{(j)} - \frac{R\overline{T}}{C_v} \frac{a_{jj} \Delta t}{\overline{H}} \tilde{\partial} w^{(j)} = T^t - \Delta t \sum_{k=1}^{j-1} \left(\tilde{a}_{jk} \left[\overline{U} \nabla T^{(k)} + \frac{R\overline{T}}{C_v} \nabla u^{(k)} \right] - a_{jk} \frac{R\overline{T}}{C_v} \frac{1}{\overline{H}} \tilde{\partial} w^{(k)} \right)$$

$$q^{(j)} - \frac{C_p}{C_v} \frac{a_{jj} \Delta t}{\overline{H}} \tilde{\partial} w^{(j)} = q^t - \Delta t \sum_{k=1}^{j-1} \left(\tilde{a}_{jk} \left[\overline{U} \nabla q^{(k)} - \mathcal{S} \nabla u^{(k)} + \frac{C_p}{C_v} \nabla u^{(k)} \right] - a_{jk} \frac{C_p}{C_v} \frac{1}{\overline{H}} \tilde{\partial} w^{(k)} \right)$$

$$q_s^{(j)} = q_s^t - \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk} \mathcal{N} \nabla u^{(k)}$$

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$$u^{(j)} = u^{\bullet}$$

$$w^{(j)} - a_{jj}\Delta t g(\tilde{\partial} + \mathcal{I})q^{(j)} = w^{\bullet}$$

$$T^{(j)} - \frac{R\overline{T}}{C_v}\frac{a_{jj}\Delta t}{\overline{H}}\tilde{\partial} w^{(j)} = T^{\bullet}$$

$$q^{(j)} - \frac{C_p}{C_v}\frac{a_{jj}\Delta t}{\overline{H}}\tilde{\partial} w^{(j)} = q^{\bullet}$$

$$q_s^{(j)} = q_s^{\bullet}$$

$$q^{(j)} - \left(\frac{a_{jj}\Delta t c_s}{\overline{H}}\right)^2 \mathcal{L}_v \cdot q^{(j)} = q^{\bullet\bullet}$$

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w^{(j)} - a_{jj}\Delta t g(\tilde{\partial} + \mathcal{I})q^{(j)} &= w^{\bullet} \\
T^{(j)} + \frac{R\overline{T}}{C_v} a_{jj}\Delta t \left(\nabla u^{(j)} - \frac{1}{\overline{\overline{H}}}\tilde{\partial} w^{(j)} \right) &= T^{\bullet} \\
q^{(j)} - a_{jj}\Delta t \left[\mathcal{S}\nabla u^{(j)} - \frac{C_p}{C_v}\left(\nabla u^{(j)} - \frac{1}{\overline{\overline{H}}}\tilde{\partial} w^{(j)} \right) \right] &= q^{\bullet} \\
q_s^{(j)} + a_{jj}\Delta t \mathcal{N}\nabla u^{(j)} &= q_s^{\bullet}
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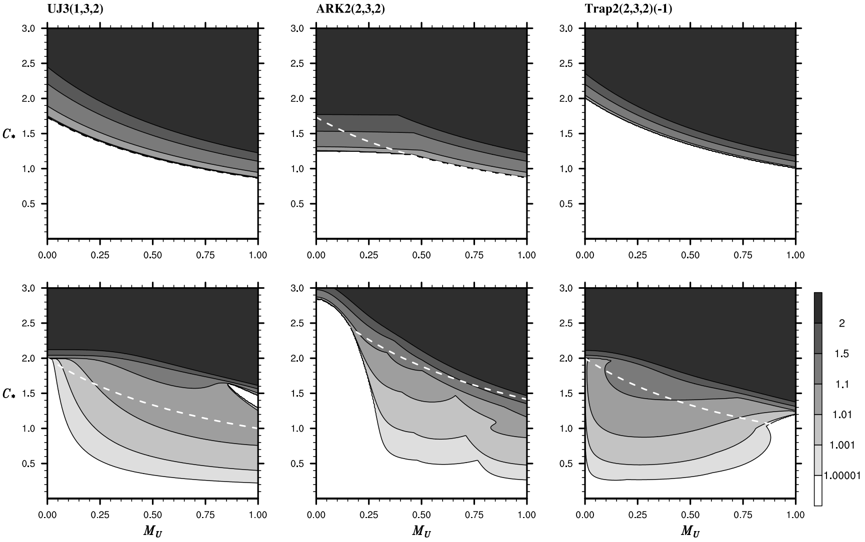
Impact of advection on the stability of RK-IMEX schemes



Colavolpe *et al.* [2016]

Mixed schemes : RK-IMEX schemes with multiple Butcher's tableaux

- Split horizontal adjustment terms in two parts, and applied own implicit Runge-Kutta scheme $\Longrightarrow$ introduce two new Butcher tableaux $\{c^u, \mathcal{A}^u, b^u\}$ and $\{c^p, \mathcal{A}^p, b^p\}$
- Have a alternative treatment explicit/implicit for each part to assure that we do not have a horizontal problem to inverse (*ie :* $a^u_{jj} a^p_{jj} = 0, \forall j \in [\![1, \nu]\!]$)
- No additional calculations

$$\partial_t X = \overbrace{\mathscr{A}(X)}^{\text{"explicit" (advection)}} + \overbrace{\mathscr{I}(X)}^{\text{"implicit"}} + \underbrace{\mathscr{U}(X)}_{\text{"implicit" (hori. mot.)}} + \underbrace{\mathscr{P}(X)}_{\text{"implicit" (hori. div.)}}$$

Time discretization :

$$\frac{X^{(j)} - X^t}{\Delta t} = \sum_{k=1}^{j-1} \tilde{a}_{jk} \mathscr{A}(X^{(k)}) + \sum_{k=1}^{j} a_{jk} \mathscr{I}(X^{(k)}) + \sum_{k=1}^{j} a^u_{jk} \mathscr{U}(X^{(k)}) + \sum_{k=1}^{j} a^p_{jk} \mathscr{P}(X^{(k)})$$

$$\frac{X^{t+\Delta t} - X^t}{\Delta t} = \sum_{j=1}^{\nu} \tilde{b}_j \mathscr{A}(X^{(j)}) + \sum_{j=1}^{\nu} b_j \mathscr{I}(X^{(j)}) + \sum_{j=1}^{\nu} b^u_j \mathscr{U}(X^{(j)}) + \sum_{j=1}^{\nu} b^p_j \mathscr{I}(X^{(j)})$$

Generalization of RK-IMEX schemes

Considering the following PDE system :

$$y'(t) = \sum_{n=1}^{N} \mathcal{F}_n\left(t, y(t)\right)$$

and the N Butcher's tableaux $\left\{c^{(n)}, \mathcal{A}^{(n)}, b^{(n)}\right\}$ so that the temporal discretization is writing by :

$$\begin{aligned} y^{(j)} =& y^0 + \Delta t \sum_{n=1}^{N} \sum_{i=1}^{\nu} a_{ji}^{(n)} \mathcal{F}_n\left(t + c_i^{(n)} \Delta t, y^{(i)}\right) \\ y^{+} =& y^0 + \Delta t \sum_{n=1}^{N} \sum_{j=1}^{\nu} b_j^{(n)} \mathcal{F}_n\left(t + c_j^{(n)} \Delta t, y^{(j)}\right) \end{aligned}$$

First order

The scheme is first order if, and only if, the N following conditions are verify :

$$ {}^{T}b^{(n)} \cdot e = 1, \qquad \forall n \in [\![1, N]\!] $$

Second order

The scheme is first order if, and only if, it is first order and verify the $N \times (N+1)$ following conditions :

$$ {}^{T}b^{(n)} \cdot c^{(n)} = \frac{1}{2}, \qquad \forall n \in [\![1, N]\!] $$

$$ {}^{T}b^{(n)} \cdot \mathcal{A}^{(m)} \cdot e = \frac{1}{2}, \qquad \forall (n, m) \in [\![1, N]\!] \times [\![1, N]\!] $$

with $e = (\underbrace{1, 1, \cdots, 1}_{\nu \text{ fois}})$

Colavolpe *et al.* [2016]

Example of new Mixed-schemes

UJ3-Mixed

$\left\{\tilde{c}, \tilde{\mathcal{A}}, \tilde{b}\right\}$

0	0					
0	0	0				
1	0	1	0			
1/2	0	1/4	1/4	0		
1	0	1/6	1/6	2/3	0	
1	0	1/6	1/6	2/3	0	0
	0	1/6	1/6	2/3	0	0

$\{c, \mathcal{A}, b\}$

0	0					
1/2	1/2	0				
1/2	1/2	0	0			
1/2	1/2	0	0	0		
1/2	1/2	0	0	0	0	
1	1/2	0	0	0	0	1/2
1	1/2	0	0	0	0	1/2

$\{c^u, \mathcal{A}^u, b^u\}$

0	0				
1/2	1/2	0			
1/2	1/2	0	0		
1/2	0	1/4	1/4	0	
1	0	1/6	1/6	2/3	0
1	0	1/6	1/6	2/3	0
1	0	1/6	1/6	2/3	0

$\{c^p, \mathcal{A}^p, b^p\}$

0	0					
0	0	0				
1	0	1	0			
1/2	0	1/4	1/4	0		
1	0	1/6	1/6	2/3	0	
1	1/2	0	0	0	0	1/2
1	1/2	0	0	0	0	1/2

ARK2-Mixed

Trap2-Mixed

UJ3-Mixed

ARK2-Mixed

$\left\{\tilde{c}, \tilde{\mathcal{A}}, \tilde{b}\right\}$

$$\begin{array}{c|ccc} 0 & 0 & & \\ 2\gamma & 2\gamma & 0 & \\ 1 & 1-\alpha & \alpha & 0 \\ \hline & \delta & \delta & \gamma \end{array}$$

$\{c, \mathcal{A}, b\}$

$$\begin{array}{c|ccc} 0 & 0 & & \\ 2\gamma & \gamma & \gamma & \\ 1 & \delta & \delta & \gamma \\ \hline & \delta & \delta & \gamma \end{array}$$

$\{c^u, \mathcal{A}^u, b^u\}$

$$\begin{array}{c|ccc} 0 & 0 & & \\ 2\gamma & 2\gamma & 0 & \\ 1 & \delta & \delta & \gamma \\ \hline & \delta & \delta & \gamma \end{array}$$

$\{c^p, \mathcal{A}^p, b^p\}$

$$\begin{array}{c|ccc} 0 & 0 & & \\ 2\gamma & \gamma & \gamma & \\ 1 & 1-\alpha & \alpha & 0 \\ \hline & \delta & \delta & \gamma \end{array}$$

où $\alpha = 1/2 + \sqrt{2}/3$, $\delta = \sqrt{2}/4$ et $\gamma = 1 - \sqrt{2}/2$

Trap2-Mixed

UJ3-Mixed

ARK2-Mixed

Trap2-Mixed

$\left\{\tilde{c}, \tilde{A}, \tilde{b}\right\}$

$$\begin{array}{c|cccc} 0 & 0 & & & \\ 0 & 0 & 0 & & \\ 1 & 1/2 & 1/2 & 0 & \\ 1 & 1/2 & 0 & 1/2 & 0 \\ \hline & 1/2 & 0 & 1/2 & 0 \end{array}$$

$\{c, \mathcal{A}, b\}$

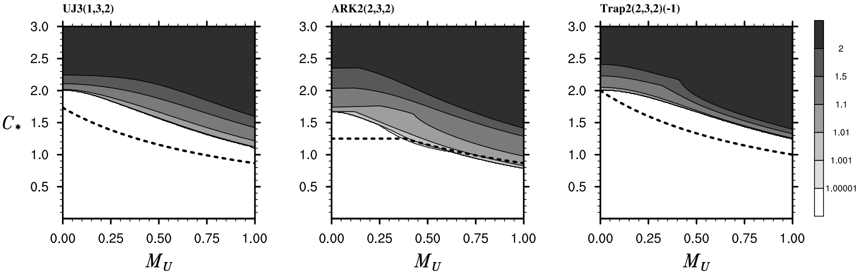
$$\begin{array}{c|cccc} 0 & 0 & & & \\ 1 & 1 & 0 & & \\ 1 & 1/2 & 0 & 1/2 & \\ 1 & 1/2 & 0 & 0 & 1/2 \\ \hline & 1/2 & 0 & 0 & 1/2 \end{array}$$

$\{c^u, \mathcal{A}^u, b^u\}$

$$\begin{array}{c|cccc} 0 & 0 & & & \\ 0 & 0 & 0 & & \\ 1 & 1/2 & 0 & 1/2 & \\ 1 & 1/2 & 0 & 1/2 & 0 \\ \hline & 1/2 & 0 & 1/2 & 0 \end{array}$$

$\{c^p, \mathcal{A}^p, b^p\}$

$$\begin{array}{c|cccc} 0 & 0 & & & \\ 1 & 1 & 0 & & \\ 1 & 1/2 & 1/2 & 0 & \\ 1 & 1/2 & 0 & 0 & 1/2 \\ \hline & 1/2 & 0 & 0 & 1/2 \end{array}$$



Properties of Mixed-schemes

- Same precision, storage factor and number of inversion than the original RK-IMEX schemes
- Better stability of Mixed-schemes than versions UFpreF (for all schemes)
- Better stability of Mixed-schemes than all other version in presence of strong advections ($M_U > 0{,}3$)

A new class of schemes which seems to be the best efficient HEVI schemes for NWP

- The use of four Runge-Kutta schemes (with three implicit) improve the stability in presence with advection, without improve the number of calculations.
- For schemes present here, the most efficient seems to be Trap2-Mixed, but it can exist more stable scheme, build as the ARK2(2,3,2).

Next stages for a operational application

- Build a model for the complete Euler's equation, and solve the vertical non-linear implicit equation with a quasi-Newton approach.
- Study this impact of the orographic terms on the stability.
- Introduce the physique on this models.