3-fold Decomposition EFB Closure for Stably Stratified Turbulence and Large-Scale Inertial Waves

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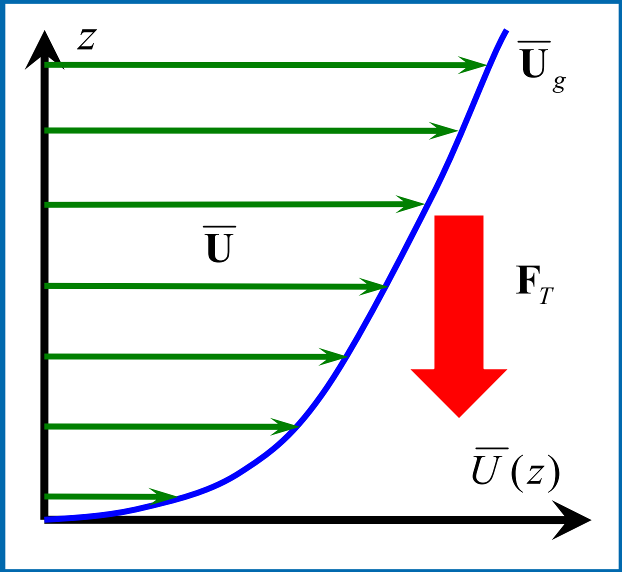
SBL Equations

$$\frac{\partial \overline{\mathbf{U}}}{\partial t} = f(\overline{\mathbf{U}} - \overline{\mathbf{U}}_g)\times\mathbf{e} - \frac{\partial \boldsymbol{\tau}}{\partial z}$$

$$\frac{\partial \overline{\Theta}}{\partial t} = -\frac{\partial F_z}{\partial z}$$

$$\frac{\partial E}{\partial t} = K_M S^2 - \left(\frac{E_K}{C_K t_T} + \frac{E_p}{C_p t_T}\right)$$



$$\vec{\boldsymbol{\tau}} = \begin{pmatrix} \langle u_x u_z \rangle \\ \langle u_y u_z \rangle \end{pmatrix} = -K_M \vec{\mathbf{S}}; \qquad f = 2\Omega \sin\phi; \qquad \overline{\mathbf{U}}_g = \frac{\vec{\nabla}_h p \times \mathbf{e}}{f}$$

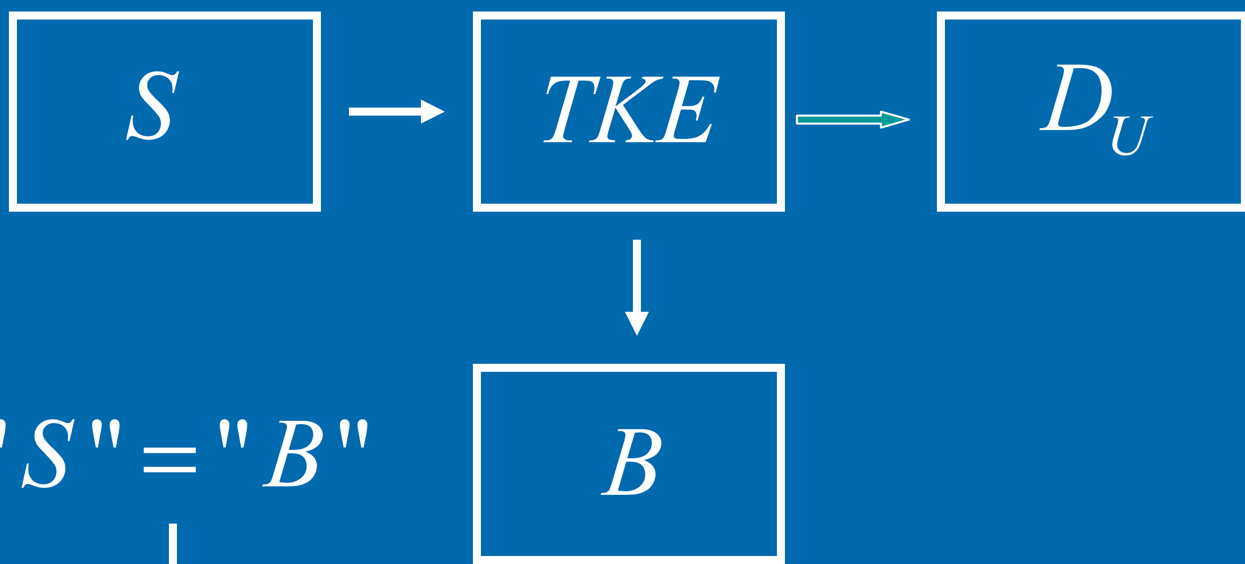
Why Turbulence?

$$\frac{\text{inertial force}}{\text{viscous force}} \propto \frac{\mathrm{v}\,l}{\nu} = \mathrm{Re} \approx 10^7 \div 10^8$$

$$\frac{\text{advective term}}{\text{diffusive term}} \propto \frac{\mathrm{v}\,l}{\kappa} = \mathrm{Pe} \approx 10^7 \div 10^8$$

Total Energy

Total Energy



$$"S" = "B"$$

↓

$$Ri_C \approx 0.25$$

$$B \propto -N^2 \left\langle u_z^2 \right\rangle$$

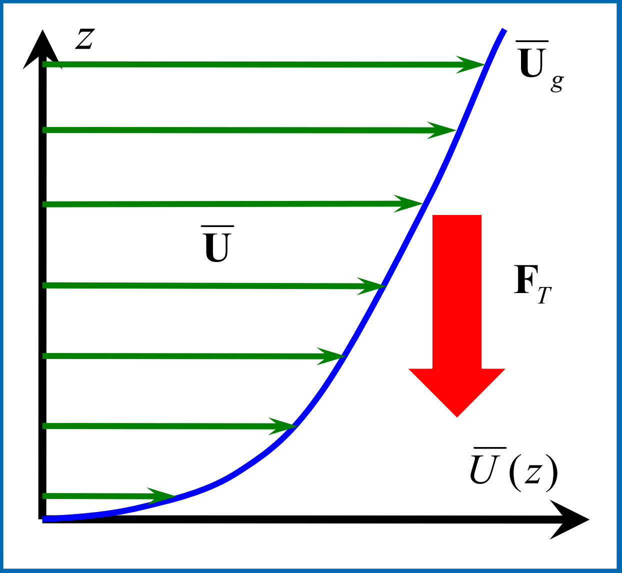
$$N^2 = \frac{g}{T}\frac{\partial \Theta}{\partial z} \equiv \beta \frac{\partial \Theta}{\partial z}$$

Total Budget Equations for SBL

$$\frac{\partial \overline{\mathbf{U}}}{\partial t} = f(\overline{\mathbf{U}} - \overline{\mathbf{U}}_g) \times \mathbf{e} - \frac{\partial \boldsymbol{\tau}}{\partial z}$$

$$\frac{\partial \overline{\Theta}}{\partial t} = -\frac{\partial F_z}{\partial z}$$

$$E_P = \frac{\beta^2}{N^2} E_\theta$$



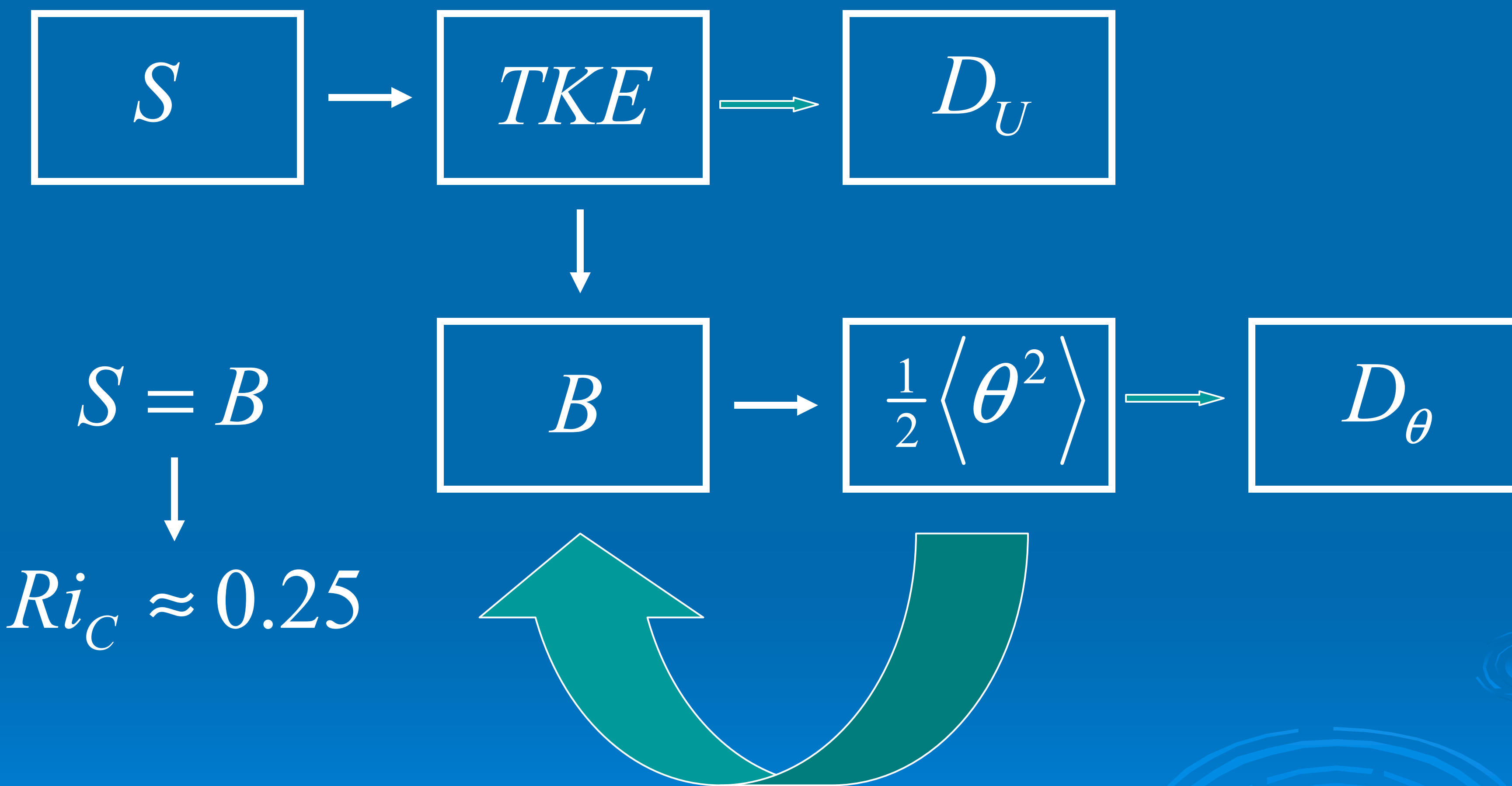
$$\frac{DE_K}{Dt} + \frac{\partial \Phi_K}{\partial z} = \Pi + \beta F_z - D_K$$

$$\frac{DE_P}{Dt} + \frac{\partial \Phi_P}{\partial z} = -\beta F_z - D_P$$

$$\Pi = K_M S^2$$

$$\frac{DF_z}{Dt} + \frac{\partial \Phi_F}{\partial z} = -D_z^F - \left\langle u_z^2 \right\rangle \frac{\partial \overline{\Theta}}{\partial z} + 2C_\theta \beta E_\theta$$

Total Energy



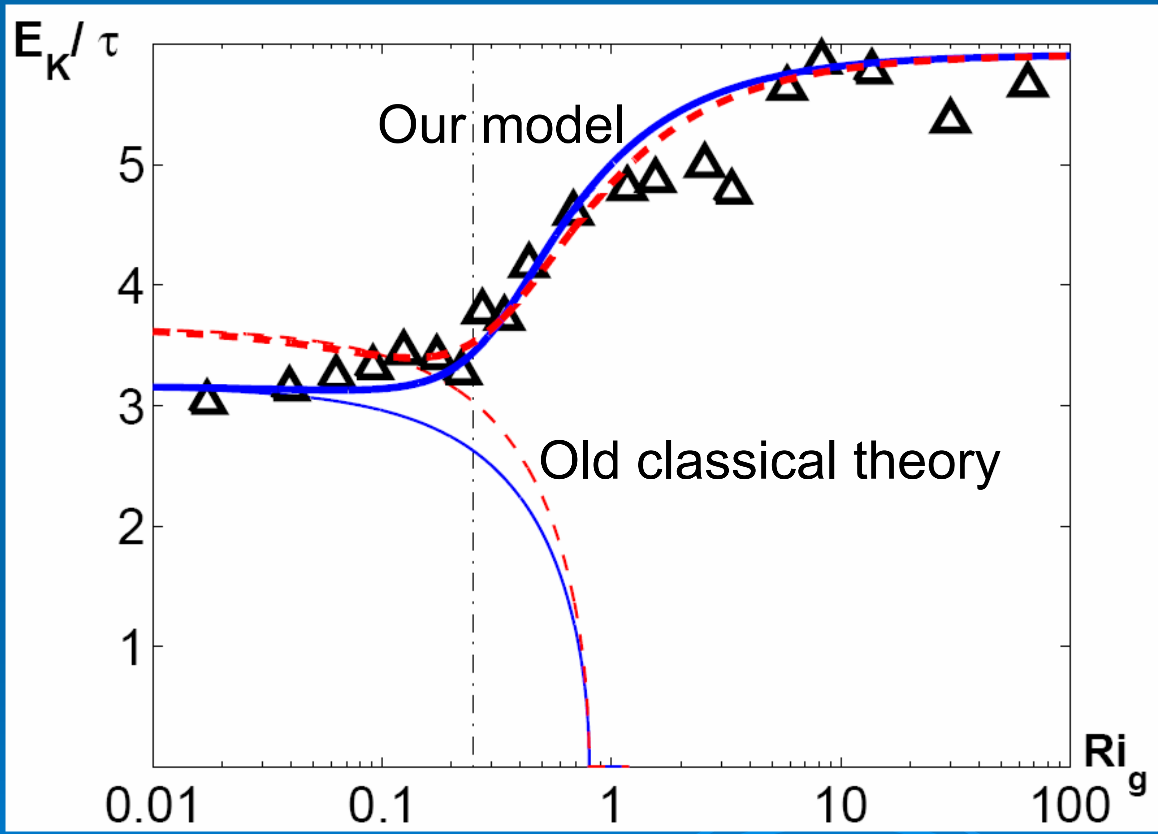
Deardorff (1970) $B \propto -N^2 \left\langle u_z^2 \right\rangle + \beta^2 \left\langle \theta^2 \right\rangle$

Steady-State Form of the Budget Equations

$$E_K = C_K t_T (\Pi + \beta F_z) \equiv C_K t_T \Pi (1 - \mathrm{Ri}_F)$$

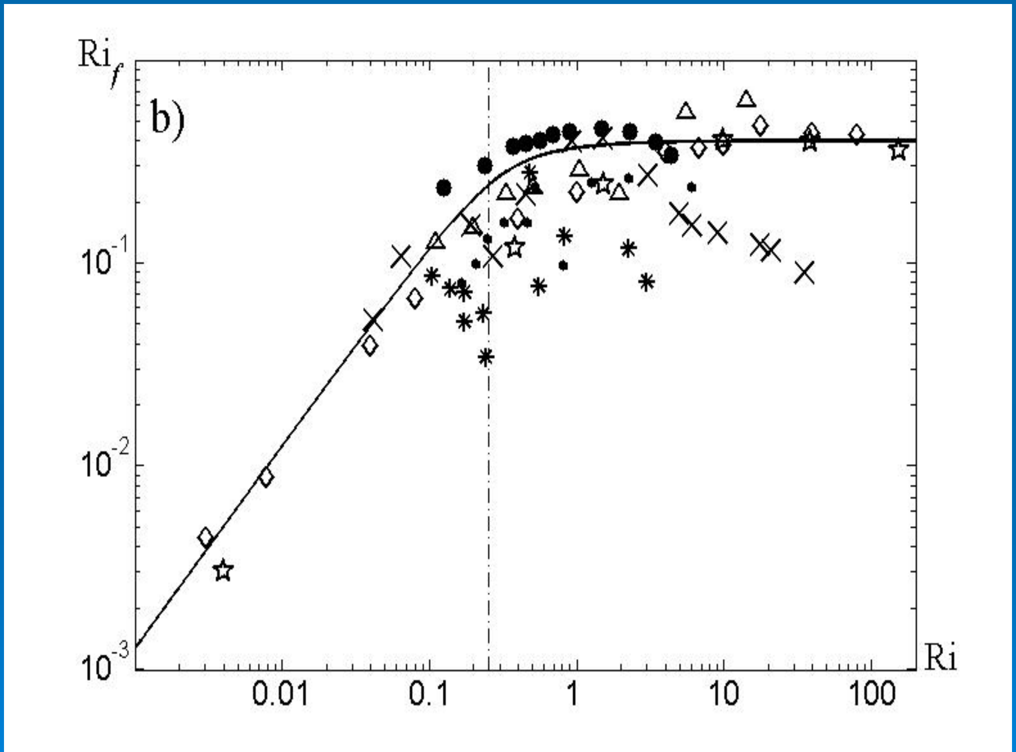
$$E_P = -C_K\, t_T\, \beta\, F_z$$

$$\mathrm{Ri}_f \equiv -\frac{\beta F_z}{\Pi} = \frac{\mathrm{Ri}}{\mathrm{Pr}_T}$$



Ri_f vs. Ri

$$\frac{1}{\mathrm{Ri}_f} = \frac{C_\tau}{C_F\,\mathrm{Ri}} + \frac{3\,C_\theta\,(1+C_r)}{C_r(1-\mathrm{Ri}_f)-3\mathrm{Ri}_f}$$



$$\mathrm{Pr}_T^{(0)} = \frac{C_\tau}{C_F}$$

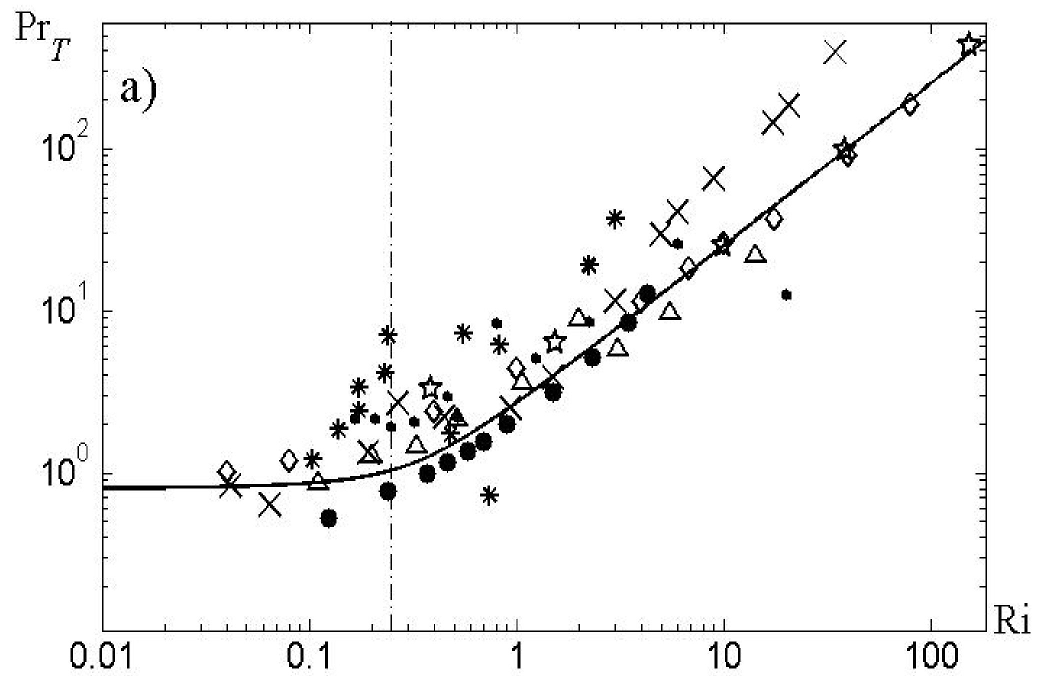
$$K_M = 2C_\tau l_z\sqrt{E_z}\,,$$

Turbulent Prandtl Number

$$\mathrm{Pr}_{\mathrm{T}}(\mathrm{Ri}) = \frac{C_\tau}{C_F} + \frac{1}{2C_K C_\tau}\frac{3(1+C_r)}{C_r}\frac{\mathrm{Ri}\, C_\theta}{1-\frac{3+C_r}{C_r}\mathrm{Ri}_f}$$

$$\mathrm{Pr}_{\mathrm{T}} \equiv \frac{K_M}{K_H}$$

$$\mathrm{Ri}_f = \frac{\mathrm{Ri}}{\mathrm{Pr}_T} \equiv -\frac{\beta F_z}{\Pi}$$



SBL Equations

$$\frac{\partial \overline{\mathbf{U}}}{\partial t} = f(\overline{\mathbf{U}} - \overline{\mathbf{U}}_g) \times \mathbf{e} - \frac{\partial \boldsymbol{\tau}}{\partial z}$$

$$\frac{\partial \overline{\Theta}}{\partial t} = -\frac{\partial F_z}{\partial z}$$

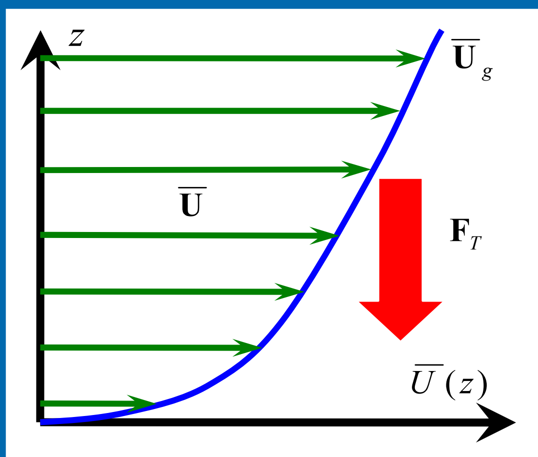
$$\frac{\partial E}{\partial t} = K_M S^2 - \left(\frac{E_K}{C_K t_T} + \frac{E_p}{C_p t_T} \right)$$

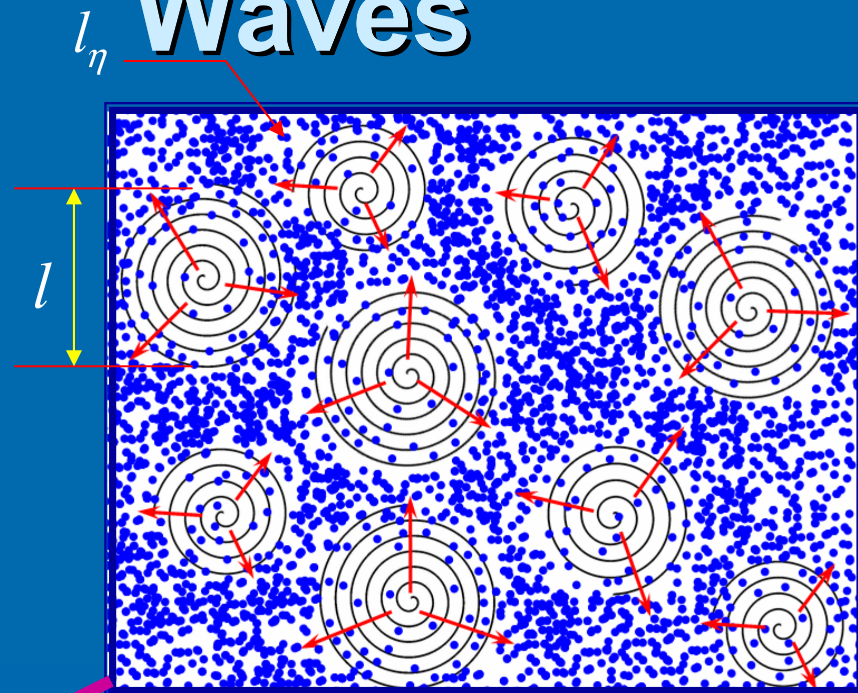
$$\vec{\boldsymbol{\tau}} = \begin{pmatrix} \langle u_x u_z \rangle \\ \langle u_y u_z \rangle \end{pmatrix} = -K_M \mathbf{S}; \qquad f = 2\Omega \sin \phi; \qquad \overline{\mathbf{U}}_g = \frac{\vec{\nabla}_h p \times \mathbf{e}}{f}$$

$$\mathbf{S} = \frac{\partial \overline{\mathbf{U}}}{\partial z} \to 0 \quad \Rightarrow \quad \mathrm{Ri} = \frac{N^2}{S^2} \to \infty$$

? ?

SBL in Presents of Gravity Waves





$$\mathbf{v} = \mathbf{U} + \mathbf{V}^W + \mathbf{u}; \quad \Theta^W = -V^W(\mathbf{k})\frac{N}{\beta}\sin(\omega t - \mathbf{k}\cdot\mathbf{r});$$

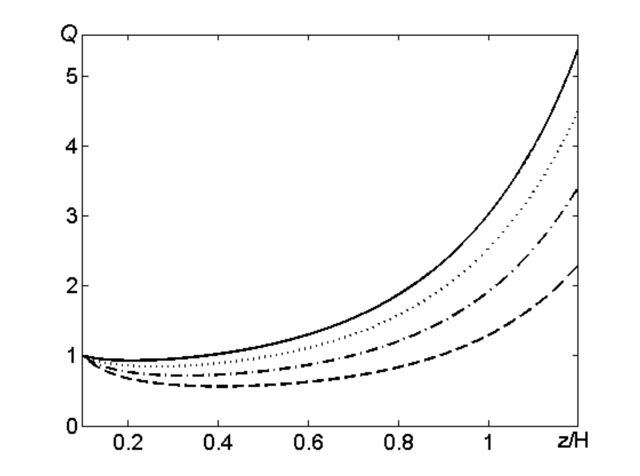
$$\mathbf{V}^W = \frac{k_h}{k} V^W(\mathbf{k})\left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2}\right)\cos(\omega t - \mathbf{k}\cdot\mathbf{r}); \; \omega = \frac{k_h}{k(z)}N(z) + \mathbf{k}\cdot\mathbf{U}$$

Total Budget Equations: BL-case in Presents of Gravity Waves

$$\frac{DE_K}{Dt} + \frac{\partial \Phi_K}{\partial z} = \Pi + \beta F_z - \frac{E_K}{C_K t_T} + \Pi^W$$

$$\frac{DE_P}{Dt} + \frac{\partial \Phi_P}{\partial z} = -\beta F_z - \frac{E_P}{C_P t_T} + \Pi_P^W$$

$$\frac{DF_z}{Dt} + \frac{\partial \Phi_F}{\partial z} = -D_z^F - \langle u_z^2 \rangle \frac{\partial \bar{\Theta}}{\partial z} + 2C_\theta \left(N^2 / \beta \right) E_p + \Pi_F^W$$



$$E_P = \frac{\beta^2}{N^2} E_\theta; \quad \Pi^W \sim G \sim \frac{E_W}{S^2 H^2} \left(\frac{H}{L_W} \right)^{3-\mu}; Q \equiv (N(z)/N_0)^2;$$

$$\mathbf{V}^W = V_0^W(\mathbf{k}) \frac{k_h}{k} \left(\mathbf{e} - \frac{(\mathbf{k} \cdot \mathbf{e}) \mathbf{k}_h}{k_h^2} \right) \cos(\omega t - \mathbf{k} \cdot \mathbf{r}); \ \omega = \frac{k_h}{k} N(z) + \mathbf{k} \cdot \mathbf{U}$$

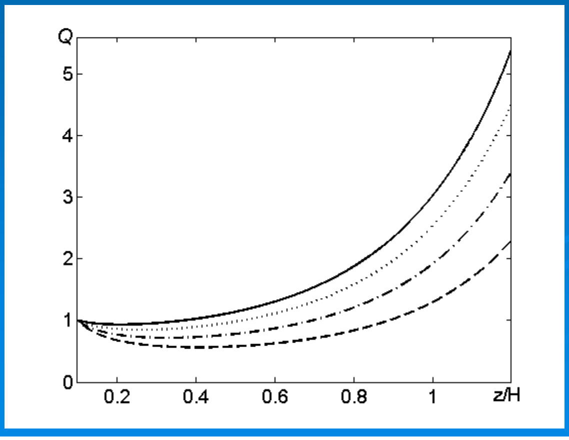
Total Budget Equations: BL-case in Presents of Gravity Waves

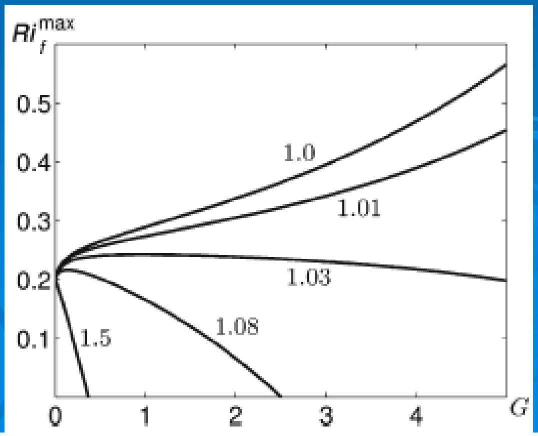
$$\frac{DE_K}{Dt} + \frac{\partial \Phi_K}{\partial z} = \Pi + \beta F_z - \frac{E_K}{C_K t_T} + \Pi^W$$

$$\frac{DE_P}{Dt} + \frac{\partial \Phi_P}{\partial z} = -\beta F_z - \frac{E_P}{C_P t_T} + \Pi_P^W \qquad \mathrm{Ri}_f \equiv -\frac{\beta F_z}{\Pi}$$

$$\frac{DF_z}{Dt} + \frac{\partial \Phi_F}{\partial z} = -D_z^F - \langle u_z^2 \rangle \frac{\partial \bar{\Theta}}{\partial z} + 2C_\theta \beta E_\theta + \Pi_F^W \quad ; Q \equiv (N(z)/N_0)^2;$$

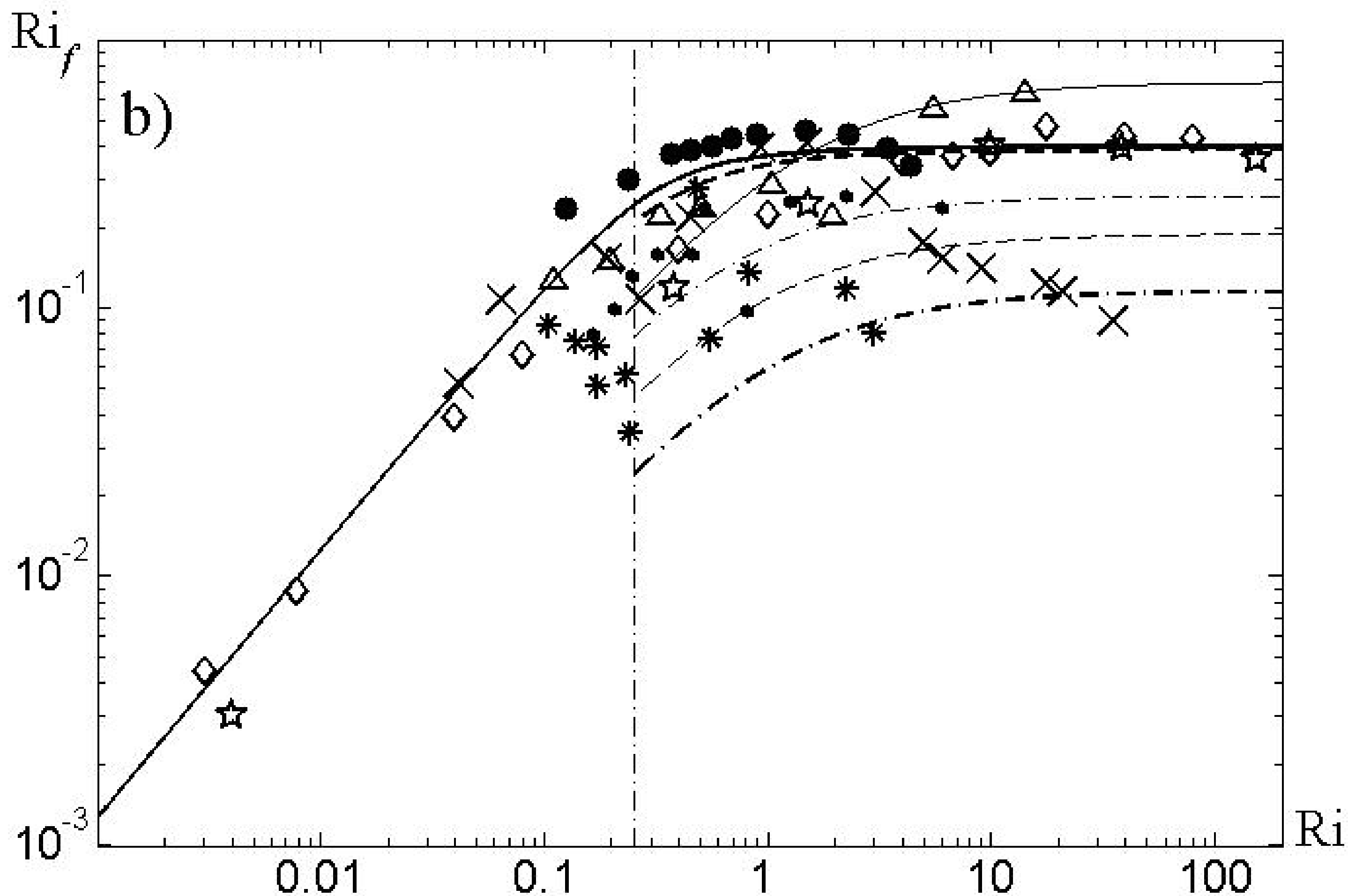
$$\Pi^W \sim G \simeq F(Q)\frac{E_W}{S^2 H^2}\left(\frac{H}{L_W}\right)^{3-\mu} \quad ; E_W = \frac{1}{4}\int \left|V_0^W(\mathbf{k})\right|^2 d\mathbf{k}$$





Ri_f vs. Ri (Waves)

$$\mathrm{Ri}_f = \frac{\mathrm{Ri}}{\mathrm{Pr}_T} \equiv -\frac{\beta F_z}{\Pi} \qquad K_M = 2C_\tau l_z \sqrt{E_z}, \qquad \mathrm{Pr}_T^{(0)} = \frac{C_\tau}{C_F}$$

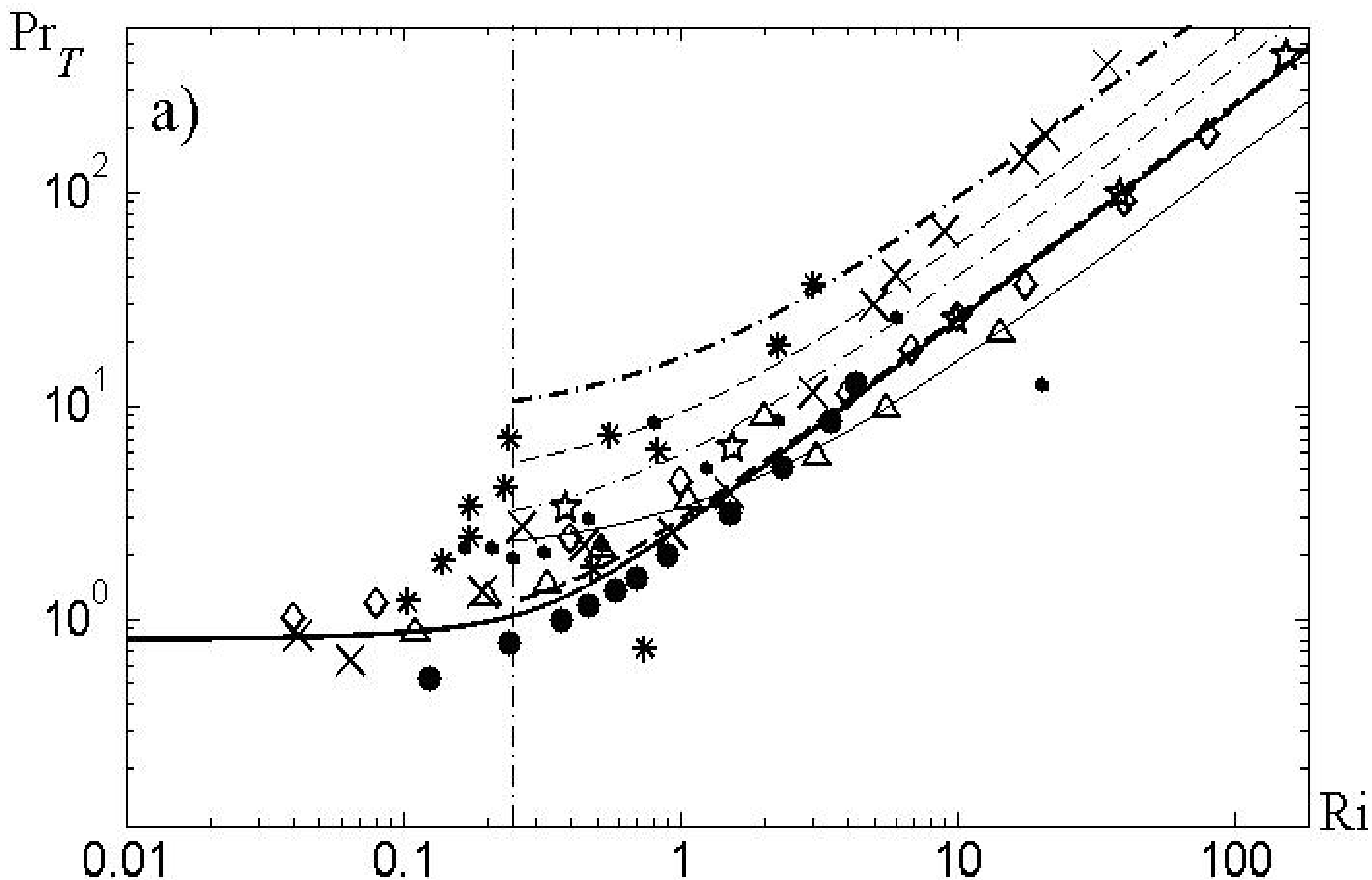


Turbulent Prandtl Number

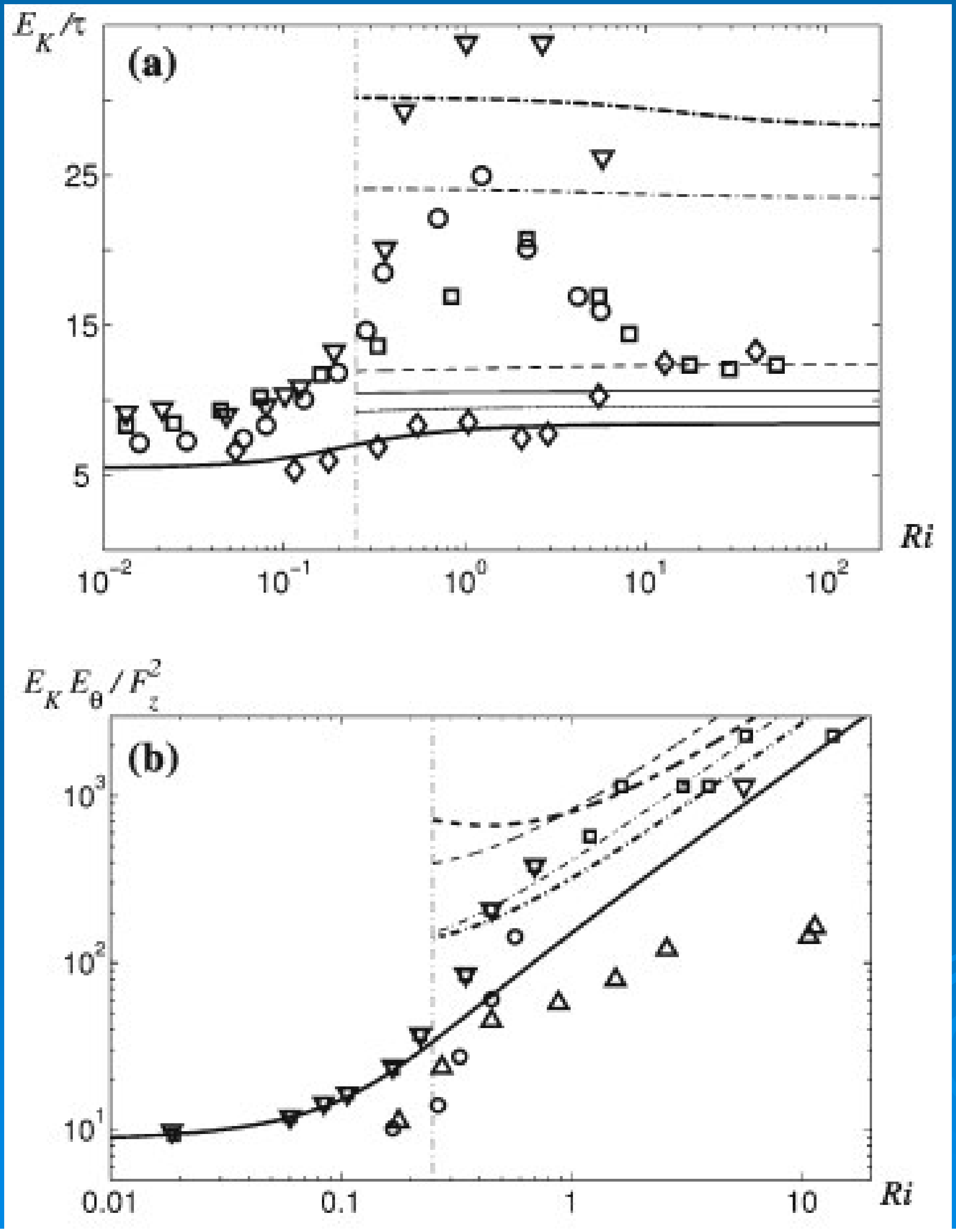
$$\mathrm{Ri}_f \equiv -\frac{\beta F_z}{\Pi}$$

$$\mathrm{Pr}_\mathrm{T} \equiv \frac{K_M}{K_H}$$

$$\mathrm{Pr}_T^{(0)} = \frac{C_\tau}{C_F}$$

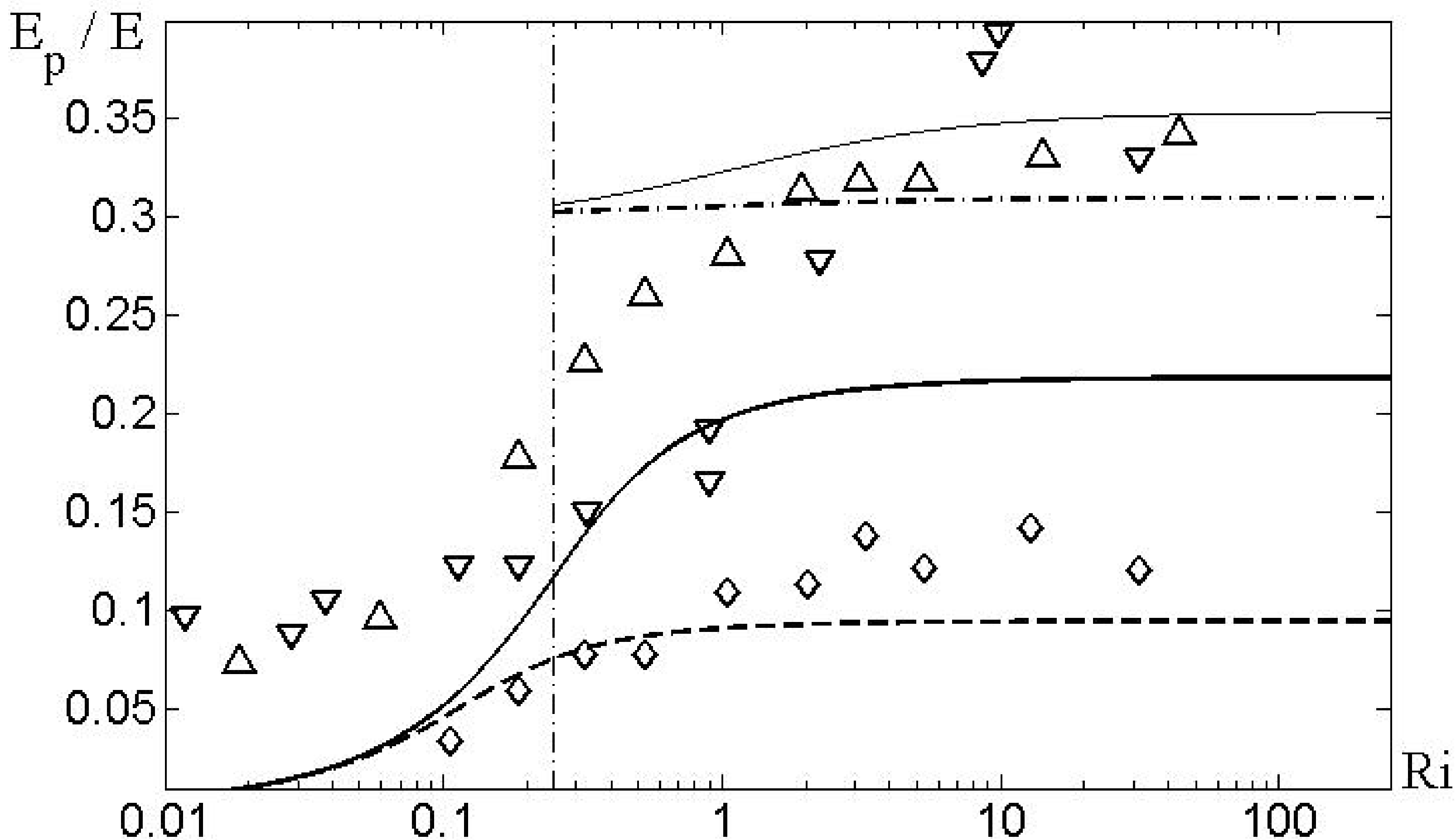


E_K, F_z vs. Ri



E_K/τ
(a)
25
15
5
Ri
10^-2
10^-1
10^0
10^1
10^2
E_K E_θ / F_z^2
(b)
10^3
10^2
10^1
0.01
0.1
1
10
Ri

$\frac{E_p}{E}$ vs. Ri (Waves)



Conclusions

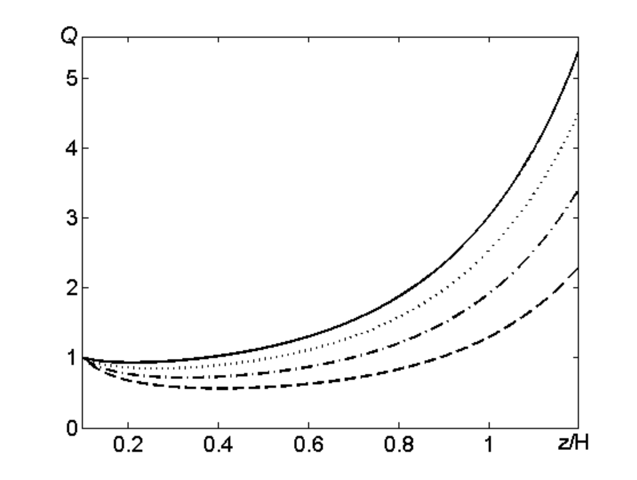
- No critical Richardson number
- Reasonable turbulent Prandtl number from theory
- Reasonable explanation of scattering of the observational data by the influence of the large-scale internal gravity waves.

Total Budget Equations: BL-case in Presents of Gravity Waves Only

$$\frac{DE_K}{Dt} + \frac{\partial \Phi_K}{\partial z} = \Pi + \beta F_z - \frac{E_K}{C_K t_T} + \Pi^W$$

$$\frac{DE_P}{Dt} + \frac{\partial \Phi_P}{\partial z} = -\beta F_z - \frac{E_P}{C_P t_T} + \Pi_P^W$$

$$\frac{DF_z}{Dt} + \frac{\partial \Phi_F}{\partial z} = -D_z^F - \langle u_z^2 \rangle \frac{\partial \bar{\Theta}}{\partial z} + 2C_\theta \left(N^2 / \beta \right) E_p + \Pi_F^W$$



$$\boxed{\mathbf{v} = \mathbf{V}^W + \mathbf{u}}; \quad \Pi^W \sim \frac{E_W}{S^2 H^2} \left(\frac{H}{L_W} \right)^{3-\mu}; Q \equiv (N(z)/N_0)^2;$$

$$\mathbf{V}^W = V_0^W(\mathbf{k}) \left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2} \right) \cos(\omega t - \mathbf{k}\cdot\mathbf{r}); \; \omega = \frac{k_h}{k} N(z) + \mathbf{k}\cdot\mathbf{U}$$

Mean field mean field equations: BL-case in Presents of Gravity Waves Only

$$\frac{DU_1}{Dt} = fU_2 - \frac{1}{\rho_0}\frac{\partial P}{\partial x} - \frac{\partial \tau_{13}}{\partial z} - \frac{\partial \tau_{1j}^{WW}}{\partial x_j}$$

$$\frac{DU_2}{Dt} = -fU_1 - \frac{1}{\rho_0}\frac{\partial P}{\partial y} - \frac{\partial \tau_{23}}{\partial z} - \frac{\partial \tau_{2j}^{WW}}{\partial x_j}$$

$$\frac{D\Theta}{Dt} = -\frac{\partial F_z}{\partial z} - \frac{\partial F_j^{WW}}{\partial x_j} + J$$

Equations for Gravity Waves

$$\frac{D\mathbf{V}^W}{Dt} = -\left(\mathbf{V}^W \cdot \nabla\right)\mathbf{U} - \nabla\left(\frac{P^W}{\rho_0}\right) + \beta\,\Theta^W \mathbf{e} - \left(\mathbf{V}^W \cdot \nabla\right)\mathbf{V}^W$$

$$\frac{D\,\Theta^W}{Dt} = -\frac{1}{\beta}\left(\mathbf{V}^W \cdot \mathbf{e}\right) N^2 - \left(\mathbf{V}^W \cdot \nabla\right)\Theta^W$$

$$\mathbf{V}^W = \frac{k_h}{k} V_0^W(\mathbf{k})\left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2}\right)\cos\left(\omega t - \mathbf{k}\cdot\mathbf{r}\right);$$

$$\Theta^W = -\frac{N}{\beta} V_0^W(\mathbf{k}) \sin(\omega t - \mathbf{k}\cdot\mathbf{r}); \qquad k(z) = \sqrt{k_z^2(z) + k_h^2} = k_0\sqrt{Q(z)}$$

$$\omega = \frac{k_h}{k(z)} N(z) + \mathbf{k}\cdot\mathbf{U}(z); \quad \frac{k_h}{k(z)} N(z) + \mathbf{k}\cdot\left(\mathbf{U}(z) - \mathbf{U}(Z_0)\right) = \frac{k_h}{k_0} N(Z_0)$$

Equations for Gravity Waves

$$\frac{D\mathbf{V}^W}{Dt} = -\left(\mathbf{V}^W \cdot \nabla\right)\mathbf{U} - \nabla\left(\frac{P^W}{\rho_0}\right) + \beta\,\Theta^W \mathbf{e} - \nabla \overleftrightarrow{\boldsymbol{\tau}}^W$$

$$\frac{D\,\Theta^W}{Dt} = -\frac{1}{\beta}\left(\mathbf{V}^W \cdot \mathbf{e}\right) N^2 - \nabla \mathbf{F}^W$$

$$\left\langle u_i u_j \right\rangle_0 = 2\begin{pmatrix} E_x & 0 & 0 \\ 0 & E_y & 0 \\ 0 & 0 & E_z \end{pmatrix}$$

$$\mathbf{V}^W = \frac{k_h}{k} V_0^W(\mathbf{k})\left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2}\right)\cos\left(\omega t - \mathbf{k}\cdot\mathbf{r}\right);$$

$$\Theta^W = -\frac{N}{\beta} V_0^W(\mathbf{k})\,\sin\left(\omega t - \mathbf{k}\cdot\mathbf{r}\right); \qquad k(z) = \sqrt{k_z^2(z) + k_h^2}$$

$$\tau_{ij}^W \approx -C_\tau\, t_T \left(\tau_{ik}\frac{\partial V_j^W}{\partial x_k} + \tau_{jk}\frac{\partial V_i^W}{\partial x_k}\right); \quad F_i^W \approx -C_F\, t_T\left(\tau_{ij}\frac{\partial\,\Theta^W}{\partial x_j} + \tau_{i3}^W \frac{\partial\,\Theta}{\partial z} + F_j \frac{\partial V_i^W}{\partial x_j}\right)$$

$$\tau_{ik} \approx \left\langle u_i u_j \right\rangle_0$$

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$$\frac{DE_K}{Dt} + \frac{\partial \Phi_K}{\partial z} = \beta F_z - \frac{E_K}{C_K t_T} + \Pi^W$$

$$\mathbf{v} = \mathbf{V}^W + \mathbf{u}$$

$$\frac{DE_P}{Dt} + \frac{\partial \Phi_P}{\partial z} = -\beta F_z - \frac{E_P}{C_P t_T} + \Pi_P^W$$

$$\frac{DF_z}{Dt} + \frac{\partial \Phi_F}{\partial z} = -D_z^F - \left\langle u_z^2 \right\rangle \frac{\partial \bar{\Theta}}{\partial z} + 2C_\theta \left(N^2 / \beta \right) E_p + \Pi_F^W$$

$$\Pi^W = -\left\langle \tau_{ij}^W \frac{\partial V_i^W}{\partial x_j} \right\rangle_W = \int \Pi^W(\mathbf{k}_0)\, d\,\mathbf{k}_0$$

$$\tau_{ij}^W \approx -C_\tau t_T \left(\tau_{ik} \frac{\partial V_j^W}{\partial x_k} + \tau_{ik} \frac{\partial V_i^W}{\partial x_k} \right)$$

$$\Pi^W(\mathbf{k}_0) = C_\tau t_T \underbrace{\tau_{jk}}_{2\begin{pmatrix} E_x & 0 & 0 \\ 0 & E_y & 0 \\ 0 & 0 & E_z \end{pmatrix}} k_j k_k \frac{e_W(k_0)}{\pi k_0^2}$$

$$= 2C_\tau t_T \left(E_z k_z^2(k_0) + E_x k_x^2 + E_y k_y^2 \right) \frac{e_W(k_0)}{\pi k_0^2}$$

Equations for Gravity Waves

$$\mathbf{V}^W \cdot \left| \frac{D\mathbf{V}^W}{Dt} = -\left(\mathbf{V}^W \cdot \nabla\right)\mathbf{U} - \nabla\left(\frac{P^W}{\rho_0}\right) + \beta\,\Theta^W \mathbf{e} - \nabla \overleftrightarrow{\boldsymbol{\tau}}^W \right.$$

$$\Theta^W \left| \frac{D\Theta^W}{Dt} = -\frac{1}{\beta}\left(\mathbf{V}^W \cdot \mathbf{e}\right) N^2 - \nabla \mathbf{F}^W \right.$$

$$\Pi_{tot} = \Pi^W + \Pi_P^W$$

$$\mathbf{V}^W = \frac{k_h}{k} V_0^W(\mathbf{k})\left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2}\right)\cos(\omega t - \mathbf{k}\cdot\mathbf{r});$$

$$\Theta^W = -\frac{N}{\beta} V_0^W(\mathbf{k}) \sin(\omega t - \mathbf{k}\cdot\mathbf{r}); \qquad k(z) = \sqrt{k_z^2(z) + k_h^2}$$

$$\tau_{ij}^W \approx -C_\tau t_T \left(\tau_{ik}\frac{\partial V_j^W}{\partial x_k} + \tau_{jk}\frac{\partial V_i^W}{\partial x_k}\right); \quad F_i^W \approx -C_F t_T \left(\tau_{ij}\frac{\partial \Theta^W}{\partial x_j} + \tau_{i3}^W \frac{\partial \Theta}{\partial z} + F_j \frac{\partial V_i^W}{\partial x_j}\right)$$

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$$\Pi^W = -\left\langle \tau_{ij}^W \frac{\partial V_i^W}{\partial x_j} \right\rangle_W = \int \Pi^W(\mathbf{k}_0)\, d\,\mathbf{k}_0$$

$$\tau_{ij}^W \approx -C_\tau t_T \left(\tau_{ik} \frac{\partial V_j^W}{\partial x_k} + \tau_{jk} \frac{\partial V_i^W}{\partial x_k} \right)$$

$$\mathbf{v} = \mathbf{V}^W + \mathbf{u}$$

$$\Pi^W(\mathbf{k}_0) = C_\tau t_T \tau_{jk} k_j k_k \frac{e_W(k_0)}{\pi k_0^2}$$

$$= 2C_\tau t_T \left(E_z k_z^2(k_0) + E_x k_x^2 + E_y k_y^2 \right) \frac{e_W(k_0)}{\pi k_0^2};$$

$$e_W(k_0) = (\mu - 1)\, E_W H^{-(\mu+1)} k_0^{-\mu}$$

$$\Pi^W = \frac{4C_\tau}{3H^2} \ell \sqrt{E_K} \left[1 + 3(Q-1)A_z\right] E_W \frac{\mu - 1}{3-\mu} \left(\frac{H}{L_W}\right)^{3-\mu} = \pi^W E_W \frac{\ell}{H^2} \sqrt{E_K}$$

$$\Pi_P^W = \frac{4C_F}{3H^2} \ell \sqrt{E_K} \left[1 + 3(Q-1)A_z\right] E_W \frac{\mu - 1}{3-\mu} \left(\frac{H}{L_W}\right)^{3-\mu} = \pi_P^W E_W \frac{\ell}{H^2} \sqrt{E_K}$$

$$\Pi_F^W = -\frac{16\, C_F C_\tau}{15 H^2 Q} \ell^2 \frac{N^2}{\beta} \left[1 + 3A_z \left(\frac{5}{6} Q - 1 \right) \right] E_W \frac{\mu - 1}{3-\mu} \left(\frac{H}{L_W}\right)^{3-\mu} = \pi_F^W \frac{N^2}{\beta} \frac{\ell^2}{H^2} E_W$$

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$$0 = \beta F_z - \frac{E_K}{C_K t_T} + \pi^W E_W \frac{\ell}{H^2}\sqrt{E_K}$$

$$0 = -\beta F_z - \frac{E_P}{C_P t_T} + \pi_P^W E_W \frac{\ell}{H^2}\sqrt{E_K}$$

$$0 = -\frac{F_z}{C_P t_T} - \left(\left\langle u_z^2 \right\rangle - 2C_\theta E_p + \frac{\ell^2}{H^2}\pi_F^W E_W \right)\frac{N^2}{\beta}$$

$$\mathbf{v} = \mathbf{V}^W + \mathbf{u}$$

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$$A_z \to 0$$

$$F_z = -\frac{\left\langle u_z^2 \right\rangle t_T C_F + K_H^{(W)}}{\left(1 + 2C_\theta C_P C_F t_T^2 N^2\right)} \frac{\partial \Theta}{\partial z}$$

$$K_H^{(W)} = \frac{8 C_F t_T \ell^2}{3H^2} E_W \frac{\mu - 1}{3 - \mu} \left(\frac{H}{L_W}\right)^{3-\mu} \left(\frac{2 C_F C_\tau}{5Q} - C_\theta C_P\right) > 0$$

$$C_K = 1.2; C_F = .31; C_\tau = .25; C_p \approx .92$$

$$C_\theta \le 3.3 \times 10^{-2} / Q$$

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$$A_z \to 0$$

$$E_K = \frac{4}{3} C_\tau C_K \frac{\ell^2}{H^2} \frac{\mu - 1}{3 - \mu} \left(\frac{H}{L_W} \right)^{3-\mu} \xi_1 E_W$$

$$\mathbf{v} = \mathbf{V}^W + \mathbf{u}$$

$$\xi_1 = \left[1 - \left(\mathrm{Pr}^{(0)} \right)^2 \frac{t_T^2 N_0^2}{\left(1 + 2 C_\theta C_P C_F t_T^2 N^2 \right)} \left(1 - Q \frac{5 C_\theta C_P}{2 C_F C_\tau} \right) \right] > 0$$

$$C_K = 1.2; C_F = .31; C_\tau = .25; C_p \approx .92; \mathrm{Pr}^{(0)} \simeq .8$$

$$3.27 \times 10^{-2} / Q \le C_\theta \le 3.37 \times 10^{-2} / Q; \quad t_T = \ell / \sqrt{E_K}$$

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$$A_z \to 0, N_0 \Rightarrow \infty$$

$$C_\theta = 3.32 \times 10^{-2} / Q$$

$$E_K = \frac{4}{3} C_\tau C_K \frac{\ell^2}{H^2} \frac{\mu - 1}{3 - \mu} \left(\frac{H}{L_W} \right)^{3-\mu} \xi_1 E_W$$

$$C_K = 1.2; C_F = .31; C_\tau = .25; C_p \approx .92; \mathrm{Pr}^{(0)} \simeq .8$$

$$\xi_1 = \left[1 - \left(\mathrm{Pr}^{(0)} \right)^2 \frac{t_T^2 N_0^2}{\left(1 + 2 C_\theta C_P C_F t_T^2 N^2 \right)} \left(1 - Q \frac{5 C_\theta C_P}{2 C_F C_\tau} \right) \right] = .503$$

$$t_T = \ell / \sqrt{E_K}$$

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$$F_z = -\frac{\langle u_z^2 \rangle t_T C_F + K_H^{(W)}}{\left(1 + 2C_\theta C_P C_F t_T^2 N^2\right)} \frac{\partial \Theta}{\partial z}$$

$A_z \to 0, N_0 \Rightarrow \infty$

$C_\theta = 3.32 \times 10^{-2} / Q$

$$K_H^{(W)} = \frac{16}{15Q} \frac{\ell^2}{H^2} C_F^2 C_\tau E_W t_T \frac{\mu - 1}{3 - \mu} \left(\frac{H}{L_W}\right)^{3-\mu} \left(1 - \frac{5}{2} \frac{C_\theta C_P}{C_F C_\tau} Q\right)$$

$C_K = 1.2; C_F = .31; C_\tau = .25; C_p \approx .92$

$C_\theta = 3.32 \times 10^{-2} / Q$

$$F_z \approx -2 \times 10^{-2} \frac{T_0 \ell^2}{L_W^2} \frac{E_W}{g t_T}$$

Equations for Gravity Waves

$$\mathbf{V}^W \cdot \left|\frac{D\mathbf{V}^W}{Dt} = -\left(\mathbf{V}^W \cdot \nabla\right)\mathbf{U} - \nabla\left(\frac{P^W}{\rho_0}\right) + \beta\,\Theta^W \mathbf{e} - \nabla\overset{\leftrightarrow}{\boldsymbol{\tau}}^W\right.$$

$$\Theta^W \left|\frac{D\,\Theta^W}{Dt} = -\frac{1}{\beta}\left(\mathbf{V}^W \cdot \mathbf{e}\right)N^2 - \nabla\mathbf{F}^W\right.$$

$$\Pi_{tot} = \Pi^W + \Pi_P^W$$

$$\mathbf{V}^W = V_0^W(\mathbf{k})\left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2}\right)\cos\left(\omega t - \mathbf{k}\cdot\mathbf{r}\right);$$

$$\Theta^W = -\frac{N\,k}{\beta\,k_h} V_0^W(\mathbf{k})\,\sin\left(\omega t - \mathbf{k}\cdot\mathbf{r}\right); \qquad k(z) = \sqrt{k_z^2(z) + k_h^2}$$

$$\tau_{ij}^W \approx -C_\tau\, t_T\left(\tau_{ik}\frac{\partial V_j^W}{\partial x_k} + \tau_{jk}\frac{\partial V_i^W}{\partial x_k}\right);\; F_i^W \approx -C_F\, t_T\left(\tau_{ij}\frac{\partial\,\Theta^W}{\partial x_j} + \tau_{i3}^W\frac{\partial\,\Theta}{\partial z} + F_j\frac{\partial V_i^W}{\partial x_j}\right)$$

Equations for Gravity Waves

$$\frac{DE_K^W}{Dt} + \operatorname{div}\mathbf{\Phi}^W = \left\langle \tau_{ij}^W \frac{\partial V_i^W}{\partial x_j} \right\rangle_W = -\prod^W$$

$$\frac{DE_P^W}{Dt} = \frac{\beta^2}{N^2} \left\langle F_j^W \frac{\partial \Theta^W}{\partial x_j} \right\rangle_W = -\prod_P^W$$

$$\prod_{tot} = \prod^W + \prod_P^W$$

$$\mathbf{V}^W = \frac{k_h}{k} V_0^W(\mathbf{k}) \left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2} \right) \cos(\omega t - \mathbf{k}\cdot\mathbf{r});$$

$$\Theta^W = -\frac{N}{\beta} V_0^W(\mathbf{k}) \sin(\omega t - \mathbf{k}\cdot\mathbf{r}); \qquad k(z) = \sqrt{k_z^2(z) + k_h^2}$$

$$\mathbf{\Phi}^W = \frac{1}{\rho_0} \left\langle p^W \, \mathbf{V}^W \right\rangle_W = \int \mathbf{C_g}(\mathbf{k}) \tilde{E}^W(\mathbf{k})\, d\mathbf{k}$$

Equations for Gravity Waves

$$\frac{DE_K^W}{Dt} + \operatorname{div}\mathbf{\Phi}^W = \left\langle \tau_{ij}^W \frac{\partial V_i^W}{\partial x_j} \right\rangle_W = -\prod^W - R$$

$$\frac{DE_P^W}{Dt} = \frac{\beta^2}{N^2} \left\langle F_j^W \frac{\partial \Theta^W}{\partial x_j} \right\rangle_W = -\prod_P^W$$

$$\prod_{tot} = \prod^W + \prod_P^W$$

$$\mathbf{V}^W = V_0^W(\mathbf{k}) \left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2} \right) \cos(\omega t - \mathbf{k}\cdot\mathbf{r});$$

$$\Theta^W = -\frac{N\,k}{\beta\,k_h} V_0^W(\mathbf{k}) \sin(\omega t - \mathbf{k}\cdot\mathbf{r}); \qquad k(z) = \sqrt{k_z^2(z) + k_h^2}$$

$$\mathbf{\Phi}^W = \frac{1}{\rho_0} \left\langle p^W\ \mathbf{V}^W \right\rangle_W = \int \mathbf{C_g}(\mathbf{k}) \tilde{E}^W(\mathbf{k})\, d\mathbf{k}$$

$$R = \mathbf{S}\cdot\boldsymbol{\tau}^{WW};\ \boldsymbol{\tau}^{WW} = \left\langle V_z^W \mathbf{V}_\perp^W \right\rangle; \mathbf{S} = \frac{d\,\mathbf{U}}{d\,z}$$

Equations for Gravity Waves

$$\frac{DE^W}{Dt} + \mathrm{div}\,\mathbf{\Phi}^W = -\prod_{tot} - \mathbf{S}\cdot\boldsymbol{\tau}^{WW}$$

$$\prod_{tot} = \prod^W + \prod_P^W$$

$$\mathbf{V}^W = \frac{k_h}{k} V_0^W(\mathbf{k}) \left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2} \right) \cos(\omega t - \mathbf{k}\cdot\mathbf{r});$$

$$\Theta^W = -\frac{N}{\beta} V_0^W(\mathbf{k}) \sin(\omega t - \mathbf{k}\cdot\mathbf{r}); \qquad k(z) = \sqrt{k_z^2(z) + k_h^2}$$

$$\mathbf{\Phi}^W = \frac{1}{\rho_0} \left\langle p^W \, \mathbf{V}^W \right\rangle_W = \int \mathbf{C_g}(\mathbf{k}) \tilde{E}^W(\mathbf{k})\, d\mathbf{k}$$

$$R = \mathbf{S}\cdot\boldsymbol{\tau}^{WW};\ \boldsymbol{\tau}^{WW} = \left\langle V_z^W \mathbf{V}_\perp^W \right\rangle; \mathbf{S} = \frac{d\mathbf{U}}{dz}$$

Equations for Gravity Waves

$$\frac{DE^W}{Dt} + \operatorname{div}\mathbf{\Phi}^W = -\prod_{tot} - \mathbf{S}\cdot\boldsymbol{\tau}^{WW}$$

$$\Pi_{tot} = \prod^W + \prod_P^W$$

$$C_\Gamma = \frac{2}{3}\frac{\mu-1}{3-\mu}(C_F + C_\tau)$$

$$\frac{\partial E^W}{\partial t} + \frac{\partial}{\partial z}\left[V_\Phi(Q)E^W\right] + \operatorname{div}_\perp\left(\mathbf{U}E^W\right) = -\Gamma(E^W)\,E^W$$

$$\mathbf{\Phi}^W = \frac{1}{\rho_0}\left\langle p^W\ \mathbf{V}^W\right\rangle_W = \int \mathbf{C_g}(\mathbf{k})\tilde{E}^W(\mathbf{k})\,d\mathbf{k} = \hat{\mathbf{e}}V_\Phi(Q)$$

$$V_\Phi(Q) = NH\,f(Q); \quad \Gamma(E^W) = C_\Gamma\sqrt{E_K(E^W)}\,\frac{\ell}{H^2}\left(\frac{H}{L_W}\right)^{3-\mu}$$

Equations for Gravity Waves

$$\frac{DE^W}{Dt} + \operatorname{div}\mathbf{\Phi}^W = -\prod_{tot} - \mathbf{S}\cdot\boldsymbol{\tau}^{WW}$$

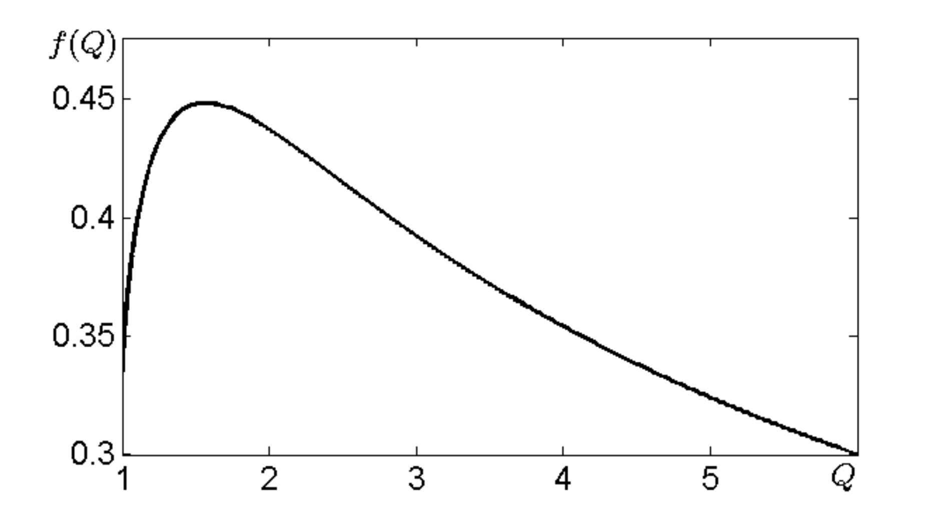
$$\Pi_{tot} = \Pi^W + \Pi_P^W$$

$$\frac{\partial E^W}{\partial t} + \frac{\partial}{\partial z}\left[V_\Phi(Q)E^W\right] + \operatorname{div}_\perp\left(\mathbf{U}E^W\right) = -\,\Gamma(E^W)\,E^W$$

$$\mathbf{\Phi}^W = \frac{1}{\rho_0}\left\langle p^W\ \mathbf{V}^W\right\rangle_W = \int \mathbf{C_g}(\mathbf{k})\tilde{E}^W(\mathbf{k})\,d\,\mathbf{k} = \hat{\mathbf{e}}V_\Phi(Q)\,E^W$$

$$V_\Phi(Q) = \frac{\mu-1}{\mu}N_0 H\,f(Q);$$

$$f(Q) = \frac{\sqrt{Q-1}}{Q}\int_0^{\frac{\pi}{2}}\sqrt{1+\frac{\cos^2\theta}{Q-1}}\ \sin^2\theta\, d\theta$$



Equations for Gravity Waves

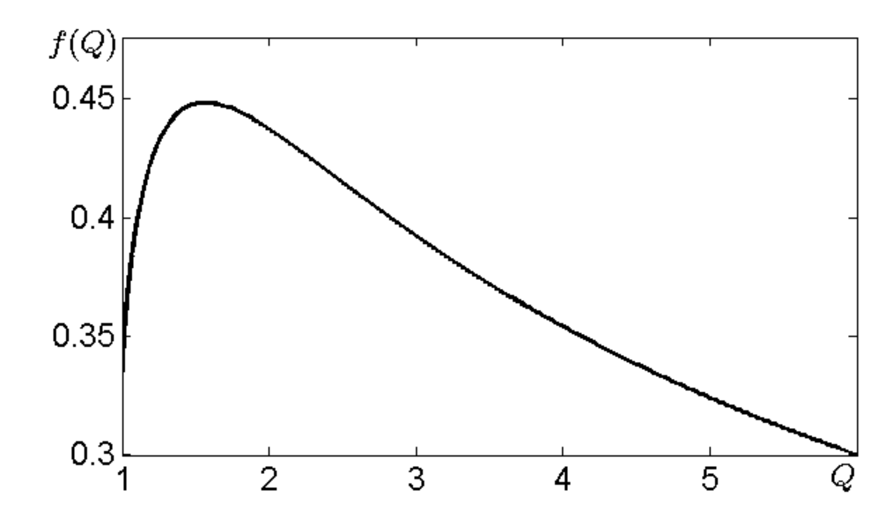
$$\frac{\partial E^W}{\partial t} + \frac{\partial}{\partial z}\left[V_\Phi(Q)E^W\right] = -\,\Gamma(E^W)\,E^W$$

$$V_\Phi(Q) = \frac{\mu - 1}{\mu} N_0 H f(Q);$$

$$f(Q) = \frac{\sqrt{Q-1}}{Q}\int_0^{\frac{\pi}{2}} \sqrt{1 + \frac{\cos^2\theta}{Q-1}}\ \sin^2\theta\, d\theta$$

$$f(1) = \frac{1}{3};$$

$$f(Q) \xrightarrow[Q\to\infty]{} \frac{\pi}{4\sqrt{Q}}$$



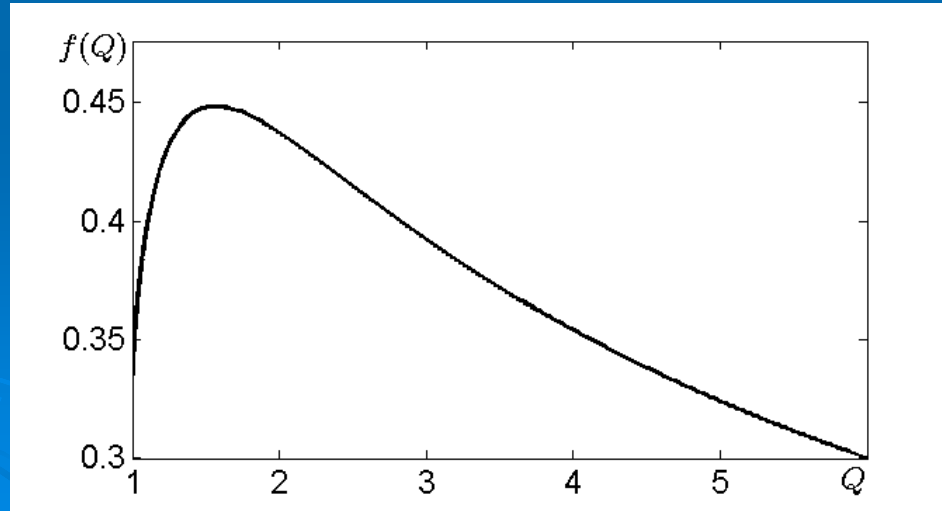
Equations for Gravity Waves

$$\frac{\partial}{\partial z}\left[V_\Phi(Q)E^W\right] = -\frac{\Gamma(E^W)}{V_\Phi(Q)}V_\Phi(Q)E^W;$$

$$\Gamma(E^W) = C_\Gamma\sqrt{E_K(E^W)}\,\frac{\ell}{H^2}\left(\frac{H}{L_W}\right)^{3-\mu};\quad V_\Phi(Q) = \frac{\mu-1}{\mu}N_0 H f(Q);\ I = V_\Phi(Q)E^W$$

$$C_\Gamma = \frac{2}{3}\frac{\mu-1}{3-\mu}(C_F + C_\tau)$$

$$E_K = \frac{4}{3}C_\tau C_K\frac{\ell^2}{H^2}\frac{\mu-1}{3-\mu}\left(\frac{H}{L_W}\right)^{3-\mu}\xi_1 E_W$$



Equations for Gravity Waves

$$I = V_\Phi(Q)E^W; \quad V_\Phi(Q) = \frac{\mu-1}{\mu} N_0 H f(Q);$$

$$\boxed{\frac{dI}{dz} = -\sigma_a I}; \qquad \sigma_a = \frac{2}{3}\frac{\mu-1}{3-\mu}(C_F + C_\tau)\sqrt{\frac{p_E}{V_\Phi^3} I}\,\frac{\ell}{H^2}\left(\frac{H}{L_W}\right)^{3-\mu};$$

$$E_K = \frac{p_E}{V_\Phi} I; \qquad p_E = \frac{4}{3} C_\tau C_K \frac{\ell^2}{H^2}\frac{\mu-1}{3-\mu}\left(\frac{H}{L_W}\right)^{3-\mu}\xi_1;$$

$$\boxed{\sigma_a = p_\sigma \sqrt{I};\ p_\sigma = \frac{4(C_F + C_\tau)\sqrt{C_\tau C_K \xi_1}}{3\sqrt{3}}\left(\frac{\mu}{N_0 H f(Q)(3-\mu)}\left(\frac{H}{L_W}\right)^{(3-\mu)}\right)^{3/2}\frac{\ell^2}{H^3}}$$

$$\boxed{\frac{dI}{dz} = -p_\sigma I^{3/2}};$$

Equations for Gravity Waves

$$I = V_\Phi(Q) E^W; \quad V_\Phi(Q) = \frac{\mu - 1}{\mu} N_0 H f(Q);$$

$$\boxed{\frac{dI}{dz} = -\sigma_a I}; \qquad \sigma_a = \frac{2}{3}\frac{\mu-1}{3-\mu}(C_F + C_\tau)\sqrt{\frac{p_E}{V_\Phi^3} I}\ \frac{\ell}{H^2}\left(\frac{H}{L_W}\right)^{3-\mu};$$

$$E_K = \frac{p_E}{V_\Phi} I; \qquad p_E = \frac{4}{3} C_\tau C_K \frac{\ell^2}{H^2}\frac{\mu-1}{3-\mu}\left(\frac{H}{L_W}\right)^{3-\mu}\xi_1;$$

Equations for Gravity Waves

$$\frac{dI}{dz} = -\sigma_a I; \qquad \sigma_a = p_\sigma \sqrt{I};$$

$$E_K = \frac{p_E}{V_\Phi} I; \qquad p_E = \frac{4}{3} C_\tau C_K \frac{\ell^2}{H^2} \frac{\mu - 1}{3 - \mu} \left(\frac{H}{L_W} \right)^{3-\mu} \xi_1;$$

$$\frac{dI}{dz} = -p_\sigma I^{3/2}$$

$$p_\sigma = \frac{4(C_F + C_\tau)\sqrt{C_\tau C_K \xi_1}}{3\sqrt{3}} \left(\frac{\mu}{N_0 H f(Q)(3-\mu)} \left(\frac{H}{L_W} \right)^{(3-\mu)} \right)^{3/2} \frac{\ell^2}{H^3}$$

Equations for Gravity Waves

$$I = V_\Phi(Q)E^W; \quad V_\Phi(Q) = \frac{\mu-1}{\mu} N_0 H f(Q); \quad \boxed{\frac{dI}{dz} = -\, p_\sigma I^{3/2}};$$

$$Q = 1 \quad I(z) = \frac{I_0}{\left(z/z_0+1\right)^2};$$

$$z_0 = \frac{2}{p_\sigma \sqrt{I_0}} = \frac{H^3}{\ell^2} \frac{3\sqrt{3}}{2(C_F + C_\tau)\sqrt{C_\tau C_K \xi_1}} \frac{N_0 H f(1)\left(3-\mu\right)}{\mu \sqrt{E_0^W}} \left(\frac{L_W}{H}\right)^{3(3-\mu)/2}$$

Equations for Gravity Waves

$$Q \geq 1 \quad I(z) = \frac{I_0}{\left(\tilde{z}/z_0 + 1\right)^2}; \qquad \tilde{z} = \int_0^z \left(\frac{f(1)}{f(Q)}\right)^{3/2} dz$$

$$z_0 = \frac{2}{p_\sigma(Q=1)\sqrt{I_0}} = H \frac{H^2}{\ell^2} \frac{3\sqrt{3}}{2(C_F + C_\tau)\sqrt{C_\tau C_K \xi_1}} \frac{N_0 H f(1)(3-\mu)}{\mu\sqrt{E_0^W}} \left(\frac{L_W}{H}\right)^{3(3-\mu)/2}$$

$N_0 \sim 3\times 10^{-3} s;$

$C_K = 1.2;$

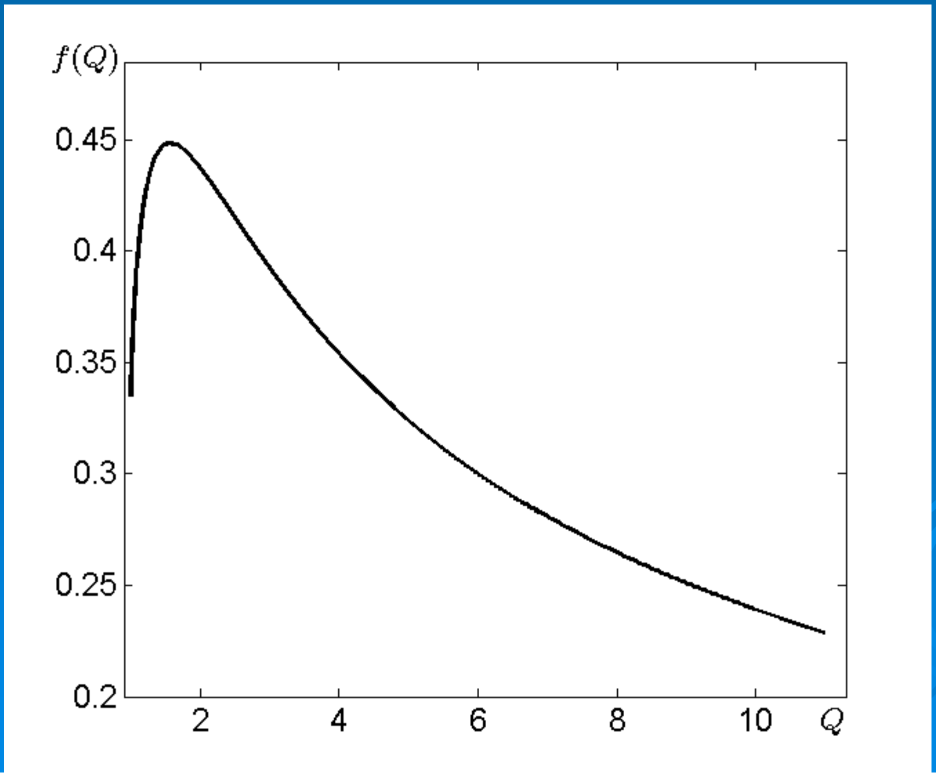
$C_\tau = .25;$

$C_F = .31;$

$H \sim 5km;$

$L_W \sim 1km;$

$\ell \sim 100m \div 1km$



$$z_0 \approx 20H \frac{N_0 H}{\sqrt{\xi_1 E_0^W}}$$

$\ell \sim L_W;\ \mu = 2;$

$N_0 H \sim .3m/s;$

$H/L_W \sim 5$

$\xi_1 \approx .5 \div 1$

Equations for Gravity Waves

$$\mathbf{V}^W = V\frac{k_h}{k}\left(\mathbf{e} - \frac{(\mathbf{k}\cdot\mathbf{e})\mathbf{k}_h}{k_h^2}\right)\exp\left[i(\omega t - \mathbf{k}\cdot\mathbf{r}) - \boldsymbol{\lambda}\cdot\mathbf{r}/2\right];$$

$$E^W = 2\frac{\mathbf{V}^W\cdot(\mathbf{V}^W)^*}{4} = \frac{V^2}{2}\exp[-2\boldsymbol{\lambda}\cdot\mathbf{r}/2]$$

$$\frac{DE^W}{Dt} + \operatorname{div}\boldsymbol{\Phi}^W = -(\nu+\kappa)k^2E^W;\ \boldsymbol{\Phi}^W = \frac{1}{\rho_0}\left\langle p^W\ \mathbf{V}^W\right\rangle_W = \mathbf{C}_\mathbf{g}(\mathbf{k})E^W$$

$$\operatorname{div}\left(\mathbf{C}_\mathbf{g}(\mathbf{k})E^W\right) = -(\nu+\kappa)k^2E^W$$

$$\mathbf{C}_\mathbf{g}(\mathbf{k})\nabla E^W = -(\nu+\kappa)k^2E^W;\mathbf{C}_g = \omega\left(\frac{\mathbf{k}}{k^2} - \frac{\mathbf{k}_h}{k_h^2}\right);\quad \omega = N\frac{k_h}{k}$$

$$\boxed{\boldsymbol{\lambda}\cdot\mathbf{C}_\mathbf{g}(\mathbf{k}) = -(\nu+\kappa)k^2}$$

Equations for Gravity Waves

$$\frac{D\mathbf{V}^W}{Dt} = -\left(\mathbf{V}^W \cdot \nabla\right)\mathbf{U} - \nabla\left(\frac{P^W}{\rho_0}\right) + \beta\,\Theta^W\mathbf{e} + \nu\Delta\mathbf{V}^W$$

$$\frac{D\,\Theta^W}{Dt} = -\frac{1}{\beta}\left(\mathbf{V}^W \cdot \mathbf{e}\right)N^2 + \kappa\Delta\,\Theta^W$$

$$\left[\Delta\left(\frac{\partial}{\partial t} - \nu\Delta\right)\left(\frac{\partial}{\partial t} - \kappa\Delta\right)\right]V_z = -N^2\Delta_\perp V_z$$

$$\mathbf{V}^W = V_z\left(\mathbf{e} - \frac{\left(\mathbf{k}\cdot\mathbf{e}\right)\mathbf{k}_h}{k_h^2}\right)\exp\left[i\left(\omega t - \tilde{\mathbf{k}}\cdot\mathbf{r}\right)\right];\; V_z = V\frac{k}{k_h}$$

Equations for Gravity Waves

$$\left[\Delta\left(\frac{\partial}{\partial t}-\nu\Delta\right)\left(\frac{\partial}{\partial t}-\kappa\Delta\right)\right]V_z=-N^2\Delta_\perp V_z$$

$$\mathbf{V}^W=V\frac{\tilde{k}_h}{\tilde{k}}\left(\mathbf{e}-\frac{\left(\tilde{\mathbf{k}}\cdot\mathbf{e}\right)\tilde{\mathbf{k}}_h}{\tilde{k}_h^2}\right)\exp\left[i\left(\omega t-\tilde{\mathbf{k}}\cdot\mathbf{r}\right)\right];$$

$$\omega^2-i\omega\left(\nu+\kappa\right)\tilde{k}^2-\nu\kappa\tilde{k}^4=N^2\frac{\tilde{k}_\perp^2}{\tilde{k}^2};$$

$$\tilde{\mathbf{k}}=\mathbf{k}-i\boldsymbol{\lambda}/2;\tilde{k}^2=k^2+\lambda^2\Rightarrow\mathbf{k}\cdot\boldsymbol{\lambda}=0\Rightarrow\boldsymbol{\lambda}\sim\mathbf{C}_g\ ?$$

$$k_z\lambda_z=-\mathbf{k}_h\cdot\boldsymbol{\lambda}_h\Rightarrow\tilde{k}_h^2=k_h^2+\lambda_h^2+ik_z\lambda_z$$

Equations for Gravity Waves

$$\frac{\omega^2}{\left(k^2+\lambda^2\right)N^2}-\frac{\nu\kappa}{N^2}\left(k^2+\lambda^2\right)=k_h^2+\lambda_h^2; \qquad \omega>\left(k^2+\lambda^2\right)\sqrt{\nu\kappa}$$

$$\omega(\nu+\kappa)\left(k^2+\lambda^2\right)=\frac{k_z\lambda_z}{\left(k^2+\lambda^2\right)}N^2 \Rightarrow \lambda_z=\frac{\omega(\nu+\kappa)\left(k^2+\lambda^2\right)^2}{k_z N^2};$$

$$\boldsymbol{\lambda}_h=-\frac{\omega(\nu+\kappa)\left(k^2+\lambda^2\right)^2}{k_h^2 N^2}\mathbf{k}_h;$$

$$\omega\gg\left(k^2+\lambda^2\right)\sqrt{\nu\kappa}$$

Equations for Gravity Waves

$$\omega \gg \left(k_0^2 + \lambda^2\right)\sqrt{\nu\kappa}$$

$$\frac{\omega^2}{\left(k_0^2+\lambda^2\right)N^2} - \cancel{\frac{\nu\kappa}{N^2}\left(k_0^2+\lambda^2\right)} = \left(\mathbf{k}_\perp^{(0)}\right)^2 + \cancel{\lambda_h^2};$$

$$\omega\left(\nu+\kappa\right)\left(k_0^2+\cancel{\lambda^2}\right) = \frac{k_z^{(0)}\lambda_z}{\left(k_0^2+\cancel{\lambda^2}\right)}N^2 \Rightarrow \lambda_z = \frac{\omega\left(\nu+\kappa\right)\left(k_0^2+\cancel{\lambda^2}\right)^2}{k_z^{(0)}N^2};$$

$$\boldsymbol{\lambda}_h = -\frac{\omega\left(\nu+\kappa\right)\left(k_0^2+\cancel{\lambda^2}\right)^2}{\left(\mathbf{k}_h^{(0)}\right)^2 N^2}\mathbf{k}_h^{(0)};$$

Equations for Gravity Waves

$$\omega \gg \left(k_0^2 + \lambda^2\right)\sqrt{\nu\kappa} \qquad \mathbf{C}_g = \omega\left(\frac{\mathbf{k}}{k^2} - \frac{\mathbf{k}_h}{k_h^2}\right)$$

$$\omega = N\frac{k_h}{k}$$

$$\omega(\nu+\kappa)k^2 = \frac{k_z^{(0)}\lambda_z}{k^2}N^2 \Rightarrow \lambda_z = \cancel{N}\,\frac{k_h}{k}\frac{(\nu+\kappa)k^4}{k_z N^{\cancel{2}}};$$

$$\boldsymbol{\lambda}_h = -N\frac{k_h}{k}\frac{(\nu+\kappa)k^4}{N^2}\frac{\mathbf{k}_h}{k_h^2} \Rightarrow \boldsymbol{\lambda} = (\nu+\kappa)k^2\frac{k_h^2}{\omega}\left[\frac{\hat{\mathbf{e}}}{k_z} - \frac{\mathbf{k}_h}{k_h^2}\right]$$

Equations for Gravity Waves

$$\boldsymbol{\lambda} = (\nu + \kappa) k^2 \frac{k_h^2}{\omega} \left(\frac{\hat{\mathbf{e}}}{k_z} - \frac{\mathbf{k}_h}{k_h^2} \right) \qquad \mathbf{C}_g = \omega \left(\frac{\mathbf{k}}{k^2} - \frac{\mathbf{k}_h}{k_h^2} \right)$$

$$\boldsymbol{\lambda} \cdot \mathbf{C}_g = (\nu + \kappa) k^2 \frac{k_h^2}{\cancel{\omega}} \left(\frac{\hat{\mathbf{e}}}{k_z} - \frac{\mathbf{k}_h}{k_h^2} \right) \cdot \cancel{\omega} \left(\frac{\mathbf{k}}{k^2} - \frac{\mathbf{k}_h}{k_h^2} \right) =$$

$$= (\nu + \kappa) k^2 k_h^2 \left(\xcancel{\frac{\hat{\mathbf{e}} \cdot \mathbf{k}}{k_z k^2}} - \xcancel{\frac{\mathbf{k}_h \cdot \mathbf{k}}{k_h^2 k^2}} + \frac{\cancel{\mathbf{k}_h \cdot \mathbf{k}_h}}{k_h^{\cancel{4}\,2}} \right) = (\nu + \kappa) k^2$$

THE END

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$$A_z \to 0$$

$$\mathbf{v} = \mathbf{V}^W + \mathbf{u}$$

$$0 = \beta F_z - \frac{E_K}{C_K t_T} + \frac{4C_\tau \ell \sqrt{E_K}}{3H^2} E_W \frac{\mu - 1}{3-\mu} \left(\frac{H}{L_W}\right)^{3-\mu}$$

$$0 = -\beta F_z - \frac{E_P}{C_P t_T} + \frac{4C_F \ell \sqrt{E_K}}{3H^2} E_W \frac{\mu - 1}{3-\mu} \left(\frac{H}{L_W}\right)^{3-\mu}$$

$$0 = -\frac{F_z}{C_F t_T} - \left\langle u_z^2 \right\rangle \frac{N^2}{\beta} + 2C_\theta E_p \frac{N^2}{\beta} - \frac{16\ell^2 C_F C_\tau}{15H^2 Q} \frac{N^2}{\beta} E_W \frac{\mu - 1}{3-\mu} \left(\frac{H}{L_W}\right)^{3-\mu}$$

Total Budget Equations: "BL-case" in Presents of Gravity Waves Only

$A_z \to 0$

$$E_P = C_P t_T \left(\frac{4 C_F \ell \sqrt{E_K}}{3H^2} E_W \frac{\mu - 1}{3 - \mu} \left(\frac{H}{L_W} \right)^{3-\mu} - \beta F_z \right)$$

$$0 = -\frac{F_z}{C_F t_T} - \left\langle u_z^2 \right\rangle \frac{N^2}{\beta} + 2 C_\theta C_P \left(\frac{4 C_F \ell^2}{3H^2} E_W \frac{\mu - 1}{3 - \mu} \left(\frac{H}{L_W} \right)^{3-\mu} - t_T \beta F_z - \frac{16 \ell^2 C_F C_\tau}{15 H^2 Q} E_W \frac{\mu - 1}{3 - \mu} \left(\frac{H}{L_W} \right)^{3-\mu} \right) \frac{N^2}{\beta}$$