12. Jozef VIVODA: "Application of the predictor-corrector method to non-hydrostatic dynamics"

Semi-implicit scheme with non-isothermal reference temperature

1. Introduction

There are three main sources of instability in the non-hydrostatic ALADIN model.

- pressure instability

  This is a weak instability due to fact that the model surface pressure differs from the reference surface pressure. The instability can be fully eliminated using correct prognostic variables ($ln(\pi_S)$ and one of $ln(p/\pi)$ or $(p-\pi)/\pi$) and σ vertical coordinate.

- thermal instability

  This instability is caused by the fact that the actual temperature varies from the reference isothermal temperature. It is apparent that full elimination for real atmospheric states is impossible because atmosphere is not isothermal. To pacify this instability in non-hydrostatic model, iterative temporal schemes have been implemented (predictor-corrector schemes).

- instability due to orography

  In spectral models, it is impossible to include orography into the reference state, because the Helmholtz solver has to be linear with constant coefficients. Orography is therefore not represented in the semi-implicit calculations. The signals from the orographic forcing are treated explicitly, if classical semi-implicit approach is used. This is not sufficient for high resolution NH models like ALADIN. Implicit treatment of the signals has to be assured by iterative temporal schemes (predictor-corrector schemes).

In real cases over areas with complex orography both critical instabilities are present, the thermal instability and the orographic one. It is not possible to separate them and to determine which kind of instability is more relevant for a given situation.

We found that the two-time-level predictor-corrector scheme is sufficient and it pacifies both kind of instabilities, with three iterations at least to achieve sufficient robustness, though we expected that one iteration would be sufficient. We could not claim which kind of instability is responsible for such behaviour.

This contribution deals with the elimination of thermal instability. A non-isothermal reference temperature is considered and consequently a new linear model is derived, for the purpose of semi-implicit calculations. According to linear analyses of stability the introduction of a non-isothermal reference temperature is sufficient to pacify the thermal instability. Structure equation is derived to determine constraints for linear vertical operators analogous to those defined in Bubnova et al., 1995.

2. Linear continuous system with non-isothermal reference temperature

An isothermal reference temperature T^* is implemented in the ALADIN model. To pacify thermal instability we assume a non-isothermal reference temperature $T^*(\eta)$. All other reference quantities remain unchanged. The reference state is considered to be at rest ($D^*= 0$, $w^* = 0$), hydrostatically balanced ($\pi^* = p^*$), with no orography ($\phi_S^* = const$) and horizontally constant ($\nabla X^* = 0$).

We apply a linearization procedure, assuming small departures from reference state. In the following, the departures are substituted by full prognostic quantities. This can be done due to the precisely chosen reference state.

For the purpose of better understanding we generalize vertical operators defined in Bubnova et al., 1995. Operators are applied on prognostic variables (X) :

$$\mathbf{G}^* X = \int_{\eta}^{1} \frac{m^*}{\pi^*} X \, d\eta$$

$$\mathbf{N}^* X = \frac{1}{\pi_s^*} \int_{0}^{1} m^* X \, d\eta \tag{1}$$

$$\mathbf{S}^* X = \frac{1}{\pi^*} \int_{0}^{\eta} m^* X \, d\eta$$

$$\mathbf{L}^* X = \partial^* (\partial^* + 1) X$$

with $\partial^* X = (\pi^*/m^*)\, \partial X/\partial\eta$. We define two new operators $\mathbf{A}^*$ and ψ^* as :

$$\psi^* X = \mathbf{G}^* T^* X = \int_{\eta}^{1} \frac{m^* T^*}{\pi^*} X \, d\eta$$

$$\mathbf{A}^* X = \mathbf{G}^* \frac{\partial T^*}{\partial \pi^*} X = \int_{\eta}^{1} \frac{m^*}{\pi^*} \cdot \frac{\partial T^*}{\partial \pi^*} X \, d\eta \tag{2}$$

We start from the system formulated with temperature (T), horizontal divergence (D), hydrostatic surface pressure (π_s), rescaled pressure departure ($\mathcal{P}_0 = (p - \pi)/\pi^*$) and pseudo vertical divergence ($d_0 = -g\, \pi^*/(m^* R T^*)\, \partial w/\partial\eta$) as prognostic variables. If we use instead of d_0 and $\mathcal{P}_0$ other "non-hydrostatic" prognostic variables based on the pressure departure (p-π) and the vertical divergence ($\partial w/\partial\eta$), then the set of linearized equations would not change and the presented results would be the same.

Applying the linearization procedure, we get linearized model equations :

$$\begin{aligned}
\frac{\partial D}{\partial t} &= -R\mathbf{G}^* \Delta T + R\psi^* \Delta \mathcal{P}_0 - RT^* \Delta \mathcal{P}_0 - R\frac{RT_s^*}{\pi_s^*} \Delta \pi_s + R\mathbf{A}^* \Delta \pi - \Delta \phi \\
\frac{\partial T}{\partial t} &= -\frac{RT^*}{C_v}(D + d_0) - \left(m^* \frac{d\eta}{dt}\right) \frac{\partial T^*}{\partial \pi^*} \\
\frac{\partial \pi_s}{\partial t} &= -\pi_s^* \mathbf{N}^* D \\
\frac{\partial \mathcal{P}_0}{\partial t} &= \mathbf{S}^* D - \frac{C_p}{C_v}(D + d_0) \\
\frac{\partial d_0}{\partial t} &= -\frac{g^2}{RT^*} \mathbf{L}^* \mathcal{P}_0
\end{aligned} \tag{3}$$

The term : $(m^* d\eta/dt)\, \partial T^*/\partial \pi^*$ represents the vertical advection of reference temperature (T^*). It can be expressed in terms of prognostic variables as :

$$\left(m^* \frac{d\eta}{dt}\right) \frac{\partial T^*}{\partial \pi^*} = -\frac{\partial T^*}{\partial \pi^*}\left(B\pi_s^* \mathbf{N}^* D - \pi^* \mathbf{S}^* D\right) \tag{4}$$

We want to eliminate from our linearized system of equations to get a structure equation. We apply operator $\partial/\partial t$ on prognostic equations for D and d_0, and then we eliminate variables T, π_s and $\mathcal{P}_0$ from our system. We get after some manipulations :

$$
\begin{aligned}
\frac{\partial^2 D}{\partial t^2} = & -R\mathbf{G}^* \cdot \Delta\left[-\frac{RT^*}{C_v}(D+d_0) - \frac{\partial T^*}{\partial \pi^*}(B\pi_s^* \mathbf{N}^* D - \pi^* \mathbf{S}^* D)\right] \\
& + R\psi^* \cdot \Delta\left[\mathbf{S}^* D - \frac{C_p}{C_v}(D+d_0)\right] - RT^* \cdot \Delta\left[\mathbf{S}^* D - \frac{C_p}{C_v}(D+d_0)\right] \\
& - \frac{RT_s^*}{\pi_s^*} \cdot \Delta\left[-\pi_s^* \mathbf{N}^* D\right] + RA^* B \cdot \Delta\left[-\pi_s^* \mathbf{N}^* D\right] \\
\frac{\partial^2 d_0}{\partial t^2} = & -\frac{g^2}{RT^*} \cdot \mathbf{L}^*\left[\mathbf{S}^* D - \frac{C_p}{C_v}(D+d_0)\right]
\end{aligned}
\tag{5}
$$

The equations can be further simplified using relationships between vertical operators :

$$
\begin{aligned}
&\mathbf{S}^* \psi^* X = \psi^* X + \mathbf{S}^* T^* X \\
&\mathbf{G}^* T^* X = \psi^* X \\
&\mathbf{G}^* \frac{\partial T^*}{\partial \pi^*} X = \mathbf{A}^* X
\end{aligned}
\tag{6}
$$

and the generalized constraint (called C1 in Bubnova et al., 1995)

$$
\text{C1:} \qquad -\mathbf{A}^* \pi^* \mathbf{S}^* X + T_s^* \mathbf{N}^* X - T^* \mathbf{S}^* X + \psi^* \mathbf{S}^* X = 0 \tag{7}
$$

After simplification we get the set of equations :

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2} - \frac{C_p}{C_v} RT^* \Delta\right) D = \left(-R\psi^* \Delta + \frac{C_p}{C_v} RT^* \Delta\right) d_0 \\
&\left(\frac{\partial^2}{\partial t^2} - \frac{g^2}{RT^*} \cdot \frac{C_p}{C_v} \mathbf{L}^*\right) d_0 = +\frac{g^2}{RT^*} \cdot \mathbf{L}^*\left(-\mathbf{S}^* + \frac{C_p}{C_v}\right) D
\end{aligned}
\tag{8}
$$

We could further define vertically dependent Laplacian speed of sound $c^2(\eta) = C_p/C_v \cdot RT^*$ and vertically dependent scale height $H(\eta) = RT^*/g$ and rewrite equations to get a more known form :

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2} - c^2 \Delta\right) D = \left(-R\psi^* + c^2\right) \Delta d_0 \\
&\left(\frac{\partial^2}{\partial t^2} - \frac{c^2}{H^2} \mathbf{L}^*\right) d_0 = +\frac{g}{H} \cdot \mathbf{L}^*\left(-\mathbf{S}^* + \frac{c^2}{gH}\right) D
\end{aligned}
\tag{9}
$$

No further elimination of variables is possible if c and H are vertically dependent. To get a structure equation average values of c and H are usually considered (Miller, 1987). This has no meaning for our purpose since it would lead to the same solution as with an isothermal reference state (the non-isothermal temperature profile would be replaced by an isothermal average temperature).

To get the solution we have to solve simultaneously the two equations (9).

We did not find an equivalent of the constraint C2 from Bubnova et al., 1995, due to fact that further elimination was not possible. One constraint C1 has been found only.

The linear Helmholtz equation for semi-implicit adjustment cannot be eliminated to one variable in the continuous context as well.

3. Linear discrete system with non-isothermal reference temperature

Our continuous domain is divided into L discrete vertical layers. Full levels are defined inside the layers (denoted as *l*, *l = 1, 2, 3,..., L*) and half levels are boundaries between layers (denoted as *l'*, *l'= 0, 1, 2, 3,..., L*).

When going from the vertically continuous to the vertically discrete case we go through the following steps (Bubnova et al., 1995) :

1. definition of discrete analogous of vertical operators with unknown vertically dependent variables
    - layer pressure thickness, $\delta\pi^*$
    - layer log-pressure thickness, δ^*
    - shift from top half-level of layer to its full level, β^*
    - shift from bottom half-level of layer to its full level, α^*

    All variables are full-level quantities. The main problem of vertical discretisation is to determine the above-mentioned unknown thicknesses and shifts.

    New problem is whether $T^*(\eta)$ shall be a full-level or half-level quantity, because T^* and its vertical derivative shall be both full-level quantities due to the new operators $\mathbf{A}^*$ and ψ^* :

$$(\psi^* X)_l = [\sum_{k=l+1}^{L} \delta_k^* T_k^* X_k + \alpha_l^* T_l^* X_l]$$

$$(\mathbf{A}^* X)_l = [\sum_{k=l+1}^{L} \delta_k^* (\frac{\partial T^*}{\partial \pi_s})_k X_k + \alpha_l^* (\frac{\partial T^*}{\partial \pi_s})_l X_l] \qquad (10)$$

    We decided to put T^* on half levels. The vertical derivative of T^* is then naturally a full-level quantity. Full-level T_l^* is then interpolated from half levels using the relationship :

$$T_l^* = (1 - \frac{\alpha_l^*}{\delta_l^*}) T_{l'}^* + \frac{\alpha_l^*}{\delta_l^*} T_{l'-1}^* \qquad (11)$$

2. discretization of the linear continuous system using discrete vertical operators
3. elimination to get a discrete analogous of (9).

    This is possible only if the discrete operators satisfy the discrete equivalent of constraint C1. But there are more unknowns than degrees of freedom in problem C1. Therefore we are not able to determine analytically all the unknown variables. This is not valid for the isothermal problem where also constraint C2 has to be fulfilled and the problem is over-determined.

    We looked for a solution using the software package *Mathematica*. We did not find any reasonable analytical solution to the full problem defined by constraint C1, even with 5 layers, which is the minimum number of layers to describe the full problem. Therefore we had to further linearize constraint C1 around average reference temperature. For this simplified C1, we've got solution of thicknesses and shifts exactly the same as for isothermal case. Therefore we decided to adopt the complete solution of unknown variables from Bubnova, 1995. The adopted thicknesses and shifts are first-order approximations of the analytical solution to the full non-isothermal constraint C1 (7).

As we already mentioned, full elimination of variables is not possible if a non-isothermal reference temperature is assumed. This becomes valid for discrete systems as well. The linear semi-implicit solver with minimum dimension 2L has to be solved in spectral space for each wavenumber (in isothermal case it is a linear problem with dimension L). So, semi-implicit scheme with non-isothermal reference state is less effective than existing scheme with isothermal reference state.

4. Practical implementation and idealized experiments with 2d plane version of NH ALADIN

Our main aim was to assess the importance of thermal instability for non-isothermal flows over orography. Implementing a non-isothermal reference state we could pacify thermal instability and study orographic instability that is the only one remaining.

We implemented semi-implicit scheme in such a way that we solve in spectral space the full linear system (3), with dimension 4L+1 for each wavenumber.

We have run the set of idealized experiments with the 2d plane NH version of ALADIN. We studied the importance of the thermal instability for idealized flows with various degrees of nonlinearity.

We simulated idealized flows over an Agnesi-shape obstacle defined by its height H_a and half-length L_a. The flow is vertically constant with wind speed V and the temperature profile is determined by temperature T_s and constant Brunt-Vaisala frequency N. The character of the flow depends on its :

1. linearity
   It is measured by non-dimensional number $C_L = N\,H_a\,/\,V$. Flow is linear if $C_L << 1$. The flows with $C_L \approx 1$ are very nonlinear. Linearity can be controlled by obstacle height.
2. hydrostaticity
   The measure of hydrostatism of the flow is dimensionless number $C_H = V\,/\,N\,L_a$. Only flows with $C_H << 1$ are considered to be hydrostatic. Hydrostatism can be controlled by the half-width of the obstacle.

The integration has been performed for various CFL ($CFL = k_{max} V \cdot \Delta t$) criteria. All the tests were run with isothermal and then with non-isothermal reference state.

All the following results are for non-hydrostatic flow regime with $C_H = 1$ (V = 10 ms^{-1}, N = 0.01 s^{-1} and L_a = 1000 m). Horizontal mesh size was dx = 200 m, number of vertical levels : 120, with a sponge applied on the 60 top levels.

Each experiment runs for n time-steps of length Δt. The length of integration was measured by dimensionless length of integration $t = V \cdot \Delta t \cdot n / L_a$, which represents length of average particle trajectory rescaled by horizontal half-width of Agnesi obstacle. Each experiment has been considered to be stable of integration finished at $t = 50$. This approach allows us to compare correctly integrations with various CFL. $t < 50$ means that the integration was unstable and it stopped earlier with an error.

4.1 Experiments with 2TL SL SI scheme with isothermal and non-isothermal reference state

The results in the tables hereafter were obtained using two-time-level semi-Lagrangian semi-implicit scheme. t is shown in the tables as a function of linearity of flow and used CFL.

From the table on the left we see that it is impossible to integrate stably non-isothermal idealized test with 2TL SL isothermal SI scheme ($T^* = 220\ K$). Already very linear flow regimes with small CFL are very unstable.

The table on the right shows that with 2TL SL non-isothermal SI scheme we can run stably linear cases (with maximum $C_L \approx 0.2$). We impose the reference temperature profile to be exactly the same as the actual temperature profile ($T_s^* = 293\ K$ with $N = 0.01\ s^{-1}$). There was no thermal instability in experiments. The orographic instability is therefore the primary reason for model instability.

2TL SL Isothermal SI scheme						
	Linearity					
CFL	1	0.8	0.6	0.4	0.2	0.1
3	2	2	2	2	2	2
2	2	2	2	2	2	2
1.5	2	2	2	2	2	2
1	2	2	2	2	2	2
0.75	2	2	2	2	2	2
0.25	1	1	1	1	1	1

2TL SL Non-isothermal SI scheme						
	Linearity					
CFL	1	0.8	0.6	0.4	0.2	0.1
3	2	2	2	2	8	10
2	2	2	2	2	**50**	**50**
1.5	2	2	2	4	**50**	**50**
1	2	2	2	4	**50**	**50**
0.75	2	2	2	4	**50**	**50**
0.25	1	1	1	**50**	**50**	**50**

4.2 Experiments with 2TL SL PC scheme with isothermal and non-isothermal reference state

We further study stability properties of 2TL SL predictor-corrector scheme with isothermal and non-isothermal reference state. In the following tables the dimensionless integration length is presented for scheme with one corrector.

2TL SL Isothermal PC scheme						
	Linearity					
CFL	1	0.8	0.6	0.4	0.2	0.1
3	5	6	8	9	12	14
2	5	6	8	28	34	35
1.5	5	6	8	**50**	**50**	**50**
1	5	6	8	**50**	**50**	**50**
0.75	5	6	**50**	**50**	**50**	**50**
0.25	**50**	**50**	**50**	**50**	**50**	**50**

2TL SL Non-isothermal PC scheme						
	Linearity					
CFL	1	0.8	0.6	0.4	0.2	0.1
3	x	x	x	x	x	x
2	x	x	x	28	34	35
1.5	x	x	x	**50**	**50**	**50**
1	x	x	8	**50**	**50**	**50**
0.75	4	6	**50**	**50**	**50**	**50**
0.25	**50**	**50**	**50**	**50**	**50**	**50**

The table on the left shows results for PC scheme based on isothermal reference state. Scheme with one corrector is sufficiently robust for tests with very small CFL ($CFL \leq 0.25$). Linear flows are stabilized for $CFL \leq 1.5$.

The table on the right is for PC scheme with non-isothermal reference state. "x" means that experiment has not been performed, since we expected the same results as for isothermal PC scheme. We see that the "island" of stability is the same as for isothermal PC scheme. Taking into account the fact that the thermal instability has not been present in experiments, we showed that orographic instability is the main source of model instability and the assumption of reference state non-isothermality doesn't improve the speed of PC scheme convergence.

5. Summary

In the present article we dealt with the introduction of a non-isothermal reference state into the semi-implicit part of the NH ALADIN model.

We found that it is analytically impossible to eliminate structure equation for single variable. Instead, the vertical modes have to be analysed by simultaneous solution of two linear differential equations.

We adopt the same discretisation methodology as in Bubnova et al., 1995. We found that discrete integral constraint C1 has no analytical solution for unknown layers thicknesses and shifts. Therefore we expand C1 around average temperature of non-isothermal profile. We get the same solution for unknowns as in Bubnova, 1995.

We found that there is no analogical constraint to C2, present in isothermal case (Bubnova, 1995). Therefore the problem in non-isothermal case is under-determined. To avoid this we fully adopt the solution from Bubnova, 1995.

Experimentally we found that the thermal instability plays minor role in unstable behaviour of 2TL schemes for NH ALADIN model. The orographic instability prevails for flows with $C_L \geq 0.4$. In operational model we could expect $C_L \approx 1$ (we consider atmospheric stability $N = 0.02\ s^{-1}$, wind speed $V = 20\ ms^{-1}$ and average height of obstacles $H_a = 1000\ m$). Therefore the 2TL non-iterative schemes based on a non-isothermal semi-implicit reference state cannot be exploited operationally in NH ALADIN model.

The thermal instability is important only for linear flows.

The speed of convergence of 2TL SL PC scheme is still a crucial issue to make it competitive with 3TL SL scheme. We found that introducing a non-isothermal reference state doesn't influence the speed of convergence.

We found out that the question of isothermality or non-isothermality of the reference flow is not important since it influences the thermal instability only. The best solution would be introduction of orography into the semi-implicit reference state $\phi_s{}^* = \phi_s$, but this is unfortunately impossible in a spectral model.

To solve orographic instability in 2TL schemes, we have two different approaches:

1. to move semi-implicit calculations from spectral space into gridpoint space and to include orography into the reference state. This would lead to a very complex semi-implicit problem that would have to be solved iteratively. Due to spectral calculations of horizontal derivatives, this approach would be very expensive. Such a solution is adopted for example in new UK Met' Office model.
2. to adopt an iterative scheme with existing isothermal semi-implicit solver as a kernel. This solution is currently implemented in ALADIN NH.