# WHY DO WE NEED THE NH MODEL ?

## Acoustic-gravity waves

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# INTRODUCTION

- The goal of this lecture is to illustrate the differences in wave propagation for the hydrostatic and nonhydrostatic systems.
- This has an impact on the response of models for orographic flows, as shown later by Jan Masek.
- Wave analysis also important as a basis for many numerical developments in models (SI scheme, stability studies...)

- In handbooks, the topic of waves is treated in $z$ coordinates.
- Here, we decribe wave in mass-coordinate, to stick to model formulation

# INTRODUCTION

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# Outline

# What are waves ?

- wave : oscillating perturbation around a stable equilibrium state in a medium
- linear wave : small oscillating perturbation ...

- The character and shape of linear waves depends on the choice of the stable equilibrium
- Complicated state $\longrightarrow$ dispersive propagation, non-uniform geometry etc...
- Simple state $\longrightarrow$ more regular propagation and geometry
- Hand-analysis of linear waves tractable only very simple equilibrium states.

# Simple small-scale atmospheric waves

We focus on small-scale atmospheric waves (neglect large-scale features as rotation $f$)

We focus on the simplest waves, i.e. on the simplest equilibrium state :

- resting (and stable $\Rightarrow$ hydrostatic)
- isothermal
- dry, nonrotating, ...

Classically, to further simplify, we will assume an unbounded fluid (boundaries impose further constraints on waves) in 2D medium $(x, \sigma)$
And of course we assume linear waves

# Outline

# EE system in $\sigma$ coordinate and model variables (d, $\mathcal{P}$)

$$\frac{d\mathbf{V}}{dt} = -RT\nabla q - \frac{RT}{(1+\mathcal{P})}\nabla\mathcal{P} - \left(1+\mathcal{P}+\sigma\frac{\partial\mathcal{P}}{\partial\sigma}\right)\nabla\phi$$

$$\frac{d\mathrm{d}}{dt} = -\frac{g^2(1+\mathcal{P})}{RT}\left(\sigma\frac{\partial}{\partial\sigma}\right)\left(1+\sigma\frac{\partial}{\partial\sigma}\right)\mathcal{P}$$

$$+ \quad \mathrm{d}(\nabla\cdot\mathbf{V} - D_3) + \frac{g(1+\mathcal{P})}{RT}\left[\nabla w \cdot \left(\sigma\frac{\partial\mathbf{V}}{\partial\sigma}\right)\right]$$

$$\frac{dT}{dt} = -\frac{RT}{C_v}D_3$$

$$\frac{d\mathcal{P}}{dt} = -(1+\mathcal{P})\left(\frac{C_p}{C_v}D_3 + \frac{\dot{\pi}}{\pi}\right)$$

$$\frac{\partial q}{\partial t} = -\int_0^1 (\nabla\cdot\mathbf{V} + \mathbf{V}\cdot\nabla q)\,d\sigma'$$

# Unbounded EE system in $\sigma$ coordinate

Warning : all symbols in red imply vertical integrals with references to the surface.

**Facultative part (sleeping allowed) :**

Procedure to show that this system may apply to a vertically unbounded fluid :

- notice the surface $\phi = \phi_s$ is not necessarily a material one
- if not material, $\phi = \phi_s$ is just a reference, immaterial surface
- Then for unbounded medium, $\sigma \in [0, \infty]$, and $\sigma = 1$ at reference surface
- to remove integrals, simply differentiate vertically

Possible discussion tonight for those interested !

# Unbounded EE system in $\sigma$ coordinate

We use the short-hand notation $\partial_\sigma = \sigma \frac{\partial}{\partial \sigma}$

$$\phi = R \int_\sigma^1 \frac{T}{1+\mathcal{P}} \frac{d\sigma}{\sigma} \quad \Rightarrow \quad \partial_\sigma \phi = -\frac{RT}{1+\mathcal{P}}$$

$$\frac{\dot{\pi}}{\pi} = \mathbf{V} \cdot \nabla q - \frac{1}{\sigma} \int_0^\sigma (\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla q) d\sigma$$

$$\Rightarrow \quad (1 + \partial_\sigma) \frac{\dot{\pi}}{\pi} = \partial_\sigma \mathbf{V} \cdot \nabla q - \nabla \cdot \mathbf{V}$$

# Unbounded EE system in $\sigma$ coordinate

Equations for **V** and $\mathcal{P}$ become :

$$
\begin{aligned}
\partial_\sigma \frac{d\mathbf{V}}{dt} &= -\partial_\sigma \left[ RT\boldsymbol{\nabla} q + \frac{RT}{(1+\mathcal{P})}\boldsymbol{\nabla}\mathcal{P} \right] \\
&\quad + [1 + (1+\partial_\sigma)\mathcal{P}]\, \boldsymbol{\nabla} \frac{RT}{(1+\mathcal{P})} - [\partial_\sigma(1+\partial_\sigma)\mathcal{P}]\, \boldsymbol{\nabla}\phi \\
(1+\partial_\sigma)\frac{d\mathcal{P}}{dt} &= -\frac{C_p}{C_v}(1+\partial_\sigma)\,[(1+\mathcal{P})D_3] \\
&\quad + [\boldsymbol{\nabla}\mathbf{V} - \partial_\sigma \mathbf{V}\cdot\boldsymbol{\nabla} q] - (1+\partial_\sigma)\left(\mathcal{P}\frac{\dot{\pi}}{\pi}\right)
\end{aligned}
$$

Now we linearize...

# Linearization of the system

Basic state (resting, isothermal, homegeneous...)

$$\partial_\sigma \frac{d\mathbf{V}}{dt} = -\partial_\sigma \left[ RT\boldsymbol{\nabla} q + \frac{RT}{(1+\mathcal{P})}\boldsymbol{\nabla}\mathcal{P} \right] + \left[1 + (1+\partial_\sigma)\mathcal{P}\right]\boldsymbol{\nabla}\frac{RT}{(1+\mathcal{P})} - \left[\partial_\sigma(1+\partial_\sigma)\mathcal{P}\right]\boldsymbol{\nabla}\phi$$

$$(1+\partial_\sigma)\frac{d\mathcal{P}}{dt} = -\frac{C_p}{C_v}(1+\partial_\sigma)\left[(1+\mathcal{P})D_3\right] + \left[\boldsymbol{\nabla}\mathbf{V} - \partial_\sigma \mathbf{V}\cdot\boldsymbol{\nabla} q\right] - (1+\partial_\sigma)\left(\mathcal{P}\frac{\dot{\pi}}{\pi}\right)$$

Terms in red are non-linear (neglected).

# Linearization

$$D_3 = \boldsymbol{\nabla} \cdot \mathbf{V} + \mathrm{d} + \frac{(1+\mathcal{P})}{RT} \boldsymbol{\nabla} \phi \cdot \partial_\sigma \mathbf{V} \quad \Longrightarrow \quad D_3 \rightarrow D + \mathrm{d}$$

$$(d/dt) = (\partial/\partial t) + \mathit{Advection} \quad \Longrightarrow \quad (d/dt) \rightarrow (\partial/\partial t)$$

$$\begin{aligned} \partial_\sigma \frac{\partial D}{\partial t} &= -RT^* \partial_\sigma \nabla^2 \mathcal{P} + R \nabla^2 T - RT^* \nabla^2 \mathcal{P} \\ &= R \nabla^2 T - RT^* (1 + \partial_\sigma) \nabla^2 \mathcal{P} \end{aligned}$$

$$(1 + \partial_\sigma) \frac{\partial \mathcal{P}}{\partial t} = -\frac{C_p}{C_v} (1 + \partial_\sigma) [D + d] + D$$

# Linearization (cont'd)

Finally the linearized unbounded system writes :

$$
\begin{aligned}
\partial_\sigma \frac{\partial D}{\partial t} &= R\nabla^2 T - RT^*(1+\partial_\sigma)\nabla^2\mathcal{P} \\
\gamma \frac{\partial \mathrm{d}}{\partial t} &= -\frac{g^2}{RT^*}\partial_\sigma(1+\partial_\sigma)\mathcal{P} \\
\frac{\partial T}{\partial t} &= -\frac{RT^*}{C_v}(D+d) \\
(1+\partial_\sigma)\frac{\partial \mathcal{P}}{\partial t} &= -\frac{C_p}{C_v}(1+\partial_\sigma)\,[D+d]+D
\end{aligned}
$$

where $\gamma$ is the marker of hydrostatic approximation

# Outline

# Analysis of waves

The linear system admit solutions of the form :
$\psi(x, \sigma, t) = \hat{\psi}(\sigma) \exp i(kx + \omega t)$
Hence, with "hats" dropped :

$$
\begin{aligned}
i\omega\partial_\sigma D &= -k^2 RT + k^2 RT^*(1+\partial_\sigma)\mathcal{P} \\
i\gamma\omega \mathrm{d} &= -\frac{g^2}{RT^*}\partial_\sigma(1+\partial_\sigma)\mathcal{P} \\
i\omega T &= -\frac{RT^*}{C_v}(D+d) \\
i\omega(1+\partial_\sigma)\mathcal{P} &= D - \frac{C_p}{C_v}(1+\partial_\sigma)(D+d)
\end{aligned}
$$

# Analysis of waves

Eliminating for $D$ and d :

$$-\omega^2 \partial_\sigma D = -k^2 R \left[ -\frac{RT^*}{C_v}(D+d) \right] + k^2 RT^* \left[ D - \frac{C_p}{C_v}(1+\partial_\sigma)(D+d) \right]$$

$$-\gamma\omega^2 \mathrm{d} = -\frac{g^2}{RT^*}\partial_\sigma \left[ D - \frac{C_p}{C_v}(1+\partial_\sigma)(D+d) \right]$$

i.e.

$$-\partial_\sigma \left[\omega^2 - k^2 c^2\right] D = k^2 RT^* \left[ \frac{R}{C_v} - \frac{C_p}{C_v}(1+\partial_\sigma) \right] \mathrm{d}$$

$$\left[ \gamma\omega^2 + \frac{g^2}{RT^*}\frac{C_p}{C_v}(1+\partial_\sigma)\partial_\sigma \right] \mathrm{d} = \frac{g^2}{RT^*} \left[ 1 - \frac{C_p}{C_v}(1+\partial_\sigma) \right] \partial_\sigma D$$

where $c^2 = (C_p/C_v)RT^*$

# Analysis of waves

Combining the two latter equations (after some eliminations) :

$$\left\{ \gamma\omega^4 - \omega^2 c^2 \left[ \gamma k^2 - \frac{(1+\partial_\sigma)\partial_\sigma}{H^2} \right] + k^2 N^2 c^2 \right\} \mathrm{d} = 0$$

where $N^2 = (g^2/C_p T^*)$, and $H = RT^*/g$

# Analysis of waves

Solutions have the form : $\mathrm{d}(\sigma) = \mathrm{d}_0 \sigma^{(i\nu H - 1/2)}$ with :

$$\gamma\omega^4 - c^2\left[\gamma k^2 + \nu^2 + 1/4H^2\right] + k^2 N^2 c^2 = 0$$

# Analysis of waves

**Case $\gamma = 0$ (Hydrostatic equations) :**

The 2 frequencies of solutions for a given $(k, \nu)$ geometry are :

$$\omega^2 = \frac{k^2 N^2}{\nu^2 + 1/4H^2}$$

These represent gravity waves

## Case $\gamma = 1$ (Euler equations) :

The 4 frequencies of solutions for a given $(k, \nu)$ geometry are :

$$\omega^2 = \frac{1}{2}\left[c^2\left(k^2+\nu^2+1/4H^2\right) \pm \sqrt{c^4\left(k^2+\nu^2+1/4H^2\right)^2 - 4k^2N^2c^2}\right]$$

These represent "gravity-acoustic" and "acoustic-gravity" waves

## Special case "gravity" for $(k \ll \nu c/N)$, e.g. $(k \approx \nu)$

$$\omega^2 = \frac{k^2N^2}{k^2+\nu^2+1/4H^2}$$

# Outline

# Compare H and NH "gravity" waves for $k \ll \nu c/N$

## Hydrostatic

$$\omega = \pm \frac{kN}{\sqrt{\nu^2 + 1/4H^2}}$$

$$[V_g]_x = [V_\phi]_x$$

## Nonhydrostatic

$$\omega = \pm \frac{kN}{\sqrt{k^2 + \nu^2 + 1/4H^2}}$$

$$[V_g]_x \neq [V_\phi]_x$$

## For hydrostatic systems :

- Gravity waves with aspect ratio $\approx 1$ wil be considerably distorted
- Orographic (stationary) gravity waves will not radiate energy in the right direction ($\longrightarrow$ Jan's talk)