Vertical discretization

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Introduction

- this talk will be rather theoretical, showing some problems which must be faced when one attempts to solve set of PDEs numerically

- the only practical consequence will be that in NH integration (`LNHDYN=.T.`) proper vertical discretization must be used:

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&NAMDYN
  NDLNPR=1,
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- anyway, setup will complain if you forget

- so if you are tired . . .

Equations to be discretized (1)

- basically these are Euler equations for perfect gas (multi-phasic case is not considered here), formulated in terrain following mass based η-coordinate:

$$\frac{\partial \pi}{\partial z} = -\rho g \qquad \pi_T \equiv 0$$

$$\pi(x, y, \eta, t) = A(\eta) + B(\eta)\pi_S(x, y, t)$$

$$\begin{matrix} \eta_T = 0 \\ \eta_S = 1 \end{matrix} \quad \Rightarrow \quad \begin{matrix} A(0) = 0 & B(0) = 0 \\ A(1) = 0 & B(1) = 1 \end{matrix}$$

- for simplicity, following NH prognostic variables will be used:

$$\mathcal{P} \equiv \frac{p - \pi}{\pi} \quad \text{– scaled NH pressure departure}$$

$$d \equiv \frac{\partial w}{\partial z} \quad \text{– true vertical divergence}$$

Equations to be discretized (2)

- with this choice of NH prognostic variables, system of Euler equations reads:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{RT}{\pi}\nabla\pi - \frac{RT}{1+\mathcal{P}}\nabla\mathcal{P} - \left[\frac{\partial(\pi\mathcal{P})}{\partial\pi} + 1\right]\nabla\phi + \boldsymbol{\mathcal{V}}$$

$$\frac{\mathrm{d}d}{\mathrm{d}t} = -\frac{\pi(1+\mathcal{P})}{RT}\cdot\frac{\partial}{\partial\pi}\left[g^2\frac{\partial(\pi\mathcal{P})}{\partial\pi} + g\mathcal{W}\right] - d(d+X) + Z$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{RT}{c_v}D_3 + \frac{Q}{c_v}$$

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = (1+\mathcal{P})\left[-\frac{c_p}{c_v}D_3 - \frac{\dot{\pi}}{\pi} + \frac{Q}{c_v T}\right]$$

$$\frac{\partial\pi_S}{\partial t} = -\int_0^1 \nabla\cdot(m\mathbf{v})\,\mathrm{d}\eta$$

$$\frac{\partial}{\partial\pi} = \frac{1}{m}\cdot\frac{\partial}{\partial\eta} \qquad m \equiv \frac{\partial\pi}{\partial\eta} = \frac{\mathrm{d}A}{\mathrm{d}\eta} + \frac{\mathrm{d}B}{\mathrm{d}\eta}\pi_S$$

Equations to be discretized (3)

- system is closed with following diagnostic relations:

$$
\begin{aligned}
D_3 &= \nabla \cdot \mathbf{v} + X + d \\
X &= \frac{\pi(1+\mathcal{P})}{RT} \cdot \frac{\partial \mathbf{v}}{\partial \pi} \cdot \nabla \phi \\
Z &= \frac{\pi(1+\mathcal{P})}{RT} \cdot \frac{\partial \mathbf{v}}{\partial \pi} \cdot \nabla (gw) \\
gw &= gw_S + \int_\eta^1 \frac{mRT}{\pi(1+\mathcal{P})} d \, \mathrm{d}\eta' \\
\phi &= \phi_S + \int_\eta^1 \frac{mRT}{\pi(1+\mathcal{P})} \, \mathrm{d}\eta' \\
\dot{\pi} &= \mathbf{v} \cdot \nabla \pi - \int_0^\eta \nabla \cdot (m\mathbf{v}) \, \mathrm{d}\eta' \\
m\dot{\eta} &= B \int_0^1 \nabla \cdot (m\mathbf{v}) \mathrm{d}\eta - \int_0^\eta \nabla \cdot (m\mathbf{v}) \, \mathrm{d}\eta'
\end{aligned}
$$

Basic principles of discretization

- at first glance it might seem that discretization is an easy task, just with many arbitrary choices to be done

- this is not true, such careless approach would most probably lead to unstable, in better case imprecise model

- in reality, discretization should be done in such way that it preserves as many continuous properties as possible

- this task is not trivial, since:

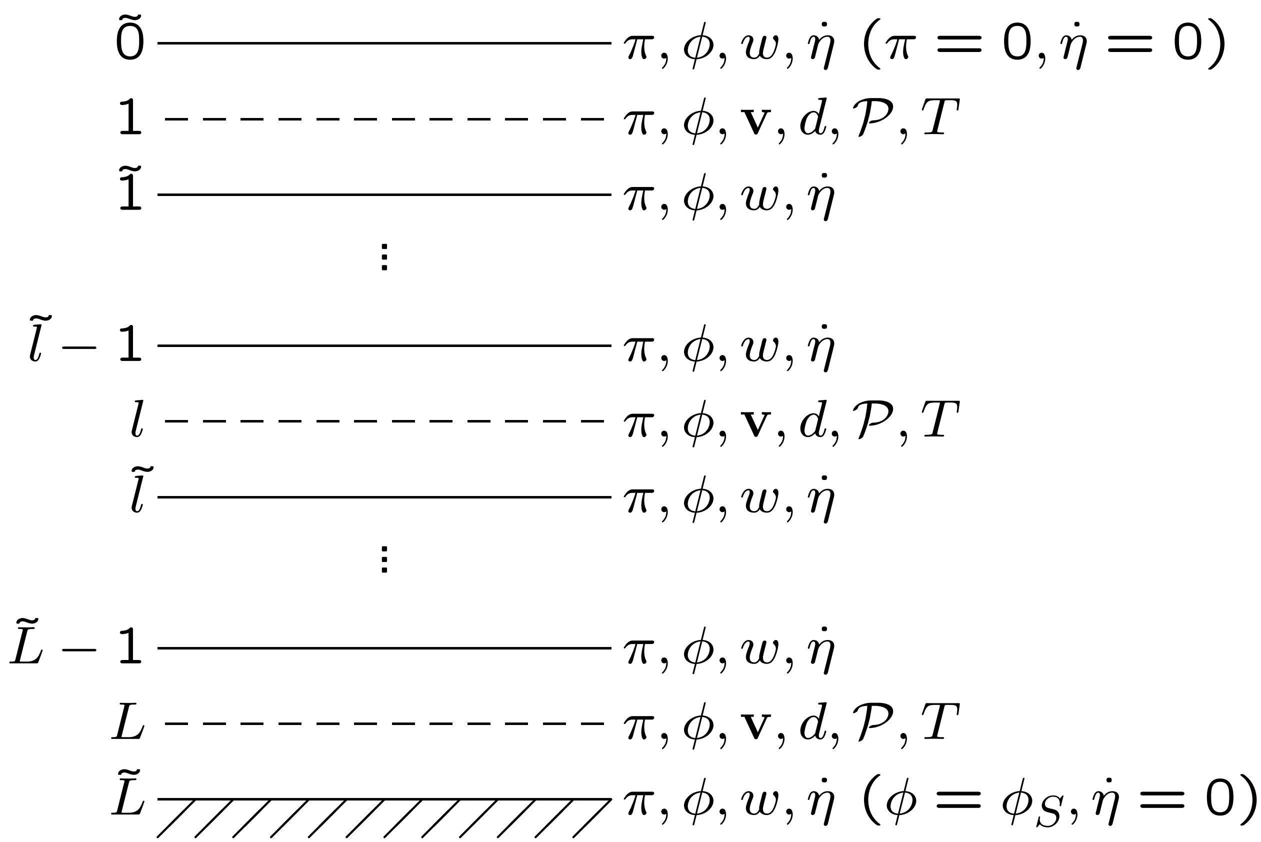
  1. no discretization scheme can preserve all continuous properties
  2. often it is not obvious which continuous properties are the important ones

Choice of vertical grid (1)

- in current ALADIN-NH, vertical discretization is done using finite difference approach
- vertically staggered grid of Lorenz type is used
- atmosphere is divided into L layers, numbered from top to bottom
- 3D prognostic variables are defined inside layers on so called full levels $1, \ldots, L$
- fluxes and vertical velocities are defined on layer interfaces or half levels $\tilde{0}, \ldots, \tilde{L}$
- half level $\tilde{0}$ is the top boundary, half level $\tilde{L}$ is the earth surface

Choice of vertical grid (2)

- schematically, vertical grid looks like this:



Introduction of continuous vertical operators

- following vertical operators can be identified in continuous system (Ψ is arbitrary function of η):

$$\mathcal{G}\Psi = \int_{\eta}^{1} \frac{m}{\pi} \Psi \, \mathrm{d}\eta'$$

$$\mathcal{S}\Psi = \frac{1}{\pi} \int_{0}^{\eta} m\Psi \, \mathrm{d}\eta'$$

$$\mathcal{N}\Psi = \frac{1}{\pi_S} \int_{0}^{1} m\Psi \, \mathrm{d}\eta$$

$$\mathcal{L}\Psi = \pi \frac{\partial^2}{\partial \pi^2}(\pi\Psi)$$

- their semi-implicit (SI) counterparts play a key role in derivation of structure equation and in elimination of variables used by Helmholtz solver

Two important properties of continuous vertical operators

- SI scheme is based on linear model which uses resting background state with flat orography, constant temperature T^* and constant surface pressure π_S^*

- for formulation of SI scheme, following two relations between continuous vertical operators are crucial:

$$\mathcal{G}^* + \mathcal{S}^* - \mathcal{G}^*\mathcal{S}^* = \mathcal{N}^* \tag{1}$$

$$\mathcal{L}^* \left[\mathcal{S}^*\mathcal{G}^* - \frac{c_p}{c_v}\mathcal{G}^* - \frac{c_p}{c_v}\mathcal{S}^* \right] = \frac{R}{c_v}\mathcal{I} \tag{2}$$

$\mathcal{I}$ - identity operator ($\mathcal{I}\Psi = \Psi$)

- these relations should hold also in discretized case

Discretized form of vertical operators (1)

- vertical functions A, B and hydrostatic pressures π are primarily defined on half levels $\tilde{l}$:

$$\pi_{\tilde{l}} = A_{\tilde{l}} + B_{\tilde{l}} \pi_{\tilde{L}}$$

- pressure difference across the layer l is given by formula:

$$\delta\pi_l = \pi_{\tilde{l}} - \pi_{\tilde{l}-1}$$

- integral operators $\mathcal{G}$, $\mathcal{S}$, $\mathcal{N}$ applied on full level quantity Ψ are discretized as:

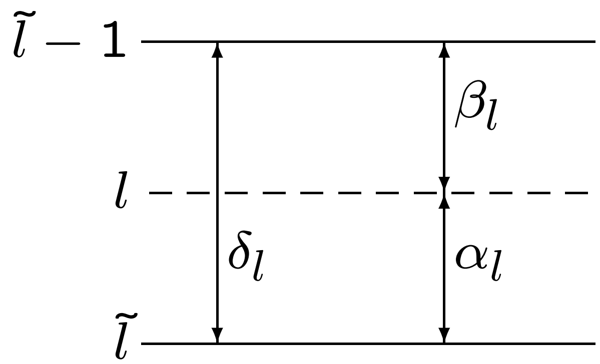
$$(\mathbf{G}\Psi)_l = \sum_{k=l+1}^{L} \Psi_k \delta_k + \Psi_l \alpha_l$$

$$(\mathbf{S}\Psi)_l = \frac{1}{\pi_l} \sum_{k=1}^{l-1} \Psi_k \delta\pi_k + \Psi_l \beta_l$$

$$(\mathbf{N}\Psi)_l = \frac{1}{\pi_{\tilde{L}}} \sum_{k=1}^{L} \Psi_k \delta\pi_k$$

Discretized form of vertical operators (2)

- symbols δ_l, α_l and β_l denote logarithmic pressure thickness of layer l, resp. its lower and upper part:



- it is not supposed that $\delta_l = \alpha_l + \beta_l$

- vertical laplacian $\mathcal{L}$ is discretized in most compact form possible, i.e. as 3-diagonal matrix:

$$(\mathbf{L}\Psi)_l = a_l \Psi_{l-1} + b_l \Psi_l + c_l \Psi_{l+1}$$

Fulfilling the constraints (1)

- the task is to find such expressions for quantities α_l^*, β_l^*, δ_l^*, π_l^*, a_l^*, b_l^*, c_l^*, which will respect constraints (1) and (2) in discretized case

- requirement that vertically discretized system respects both constraints (1) and (2) turned to be too strong, in fact it cannot be respected by any reasonable choice

- however, it is possible to satisfy weaker requirement:

$$\mathbf{G}^* + \mathbf{S}^* - \mathbf{G}^*\mathbf{S}^* = \mathbf{N}^*$$

$$\mathbf{L}^* \left[\mathbf{S}^*\mathbf{G}^* - \frac{c_p}{c_v}\mathbf{G}^* - \frac{c_p}{c_v}\mathbf{S}^* \right] = \frac{R}{c_v}\mathbf{T}^*$$

$\mathbf{T}^*$ – 3-diagonal smoothing operator (sum of each row is 1)

- appearance of smoothing operator $\mathbf{T}^*$ means that constraint (2) is fulfilled only approximately in discretized system

- anyway, $\mathbf{T}^*\Psi \rightarrow \Psi$ as vertical mesh size tends to zero

Fulfilling the constraints (2)

- in order to maintain stability, solution found for SI quantities must be extended also to non-linear model

- the procedure is straightforward (it is enough to remove stars), bottom boundary treatment for vertical laplacian $\mathbf{L}$ being the only exception

- desired results for full levels $2, \ldots, L$ are:

$$\pi_l = \sqrt{\pi_{\tilde{l}-1}\pi_{\tilde{l}}} \qquad \delta_l = \frac{\delta\pi_l}{\pi_l} \qquad \alpha_l = \beta_l = 1 - \sqrt{\frac{\pi_{\tilde{l}-1}}{\pi_{\tilde{l}}}}$$

- full level 1 is a special case:

$$\delta_1 = 2 + \frac{c_v}{R} \qquad \pi_1 = \frac{\delta\pi_1}{\delta_1} \qquad \alpha_1 = \beta_1 = 1$$

Fulfilling the constraints (3)

- vertically discretized laplacian applied on scaled NH pressure departure $\mathcal{P}$ reads ($l = 2, \ldots, L-1$):

$$(\mathbf{L}\mathcal{P})_l = \left[\pi \frac{\partial}{\partial \pi}\left(\frac{\partial(\pi \mathcal{P})}{\partial \pi}\right)\right]_l = \frac{1}{\delta_l}\left[\left(\frac{\partial(\pi \mathcal{P})}{\partial \pi}\right)_{\tilde{l}} - \left(\frac{\partial(\pi \mathcal{P})}{\partial \pi}\right)_{\tilde{l}-1}\right]$$

$$\left(\frac{\partial(\pi \mathcal{P})}{\partial \pi}\right)_{\tilde{l}} = \frac{\pi_{l+1}\mathcal{P}_{l+1} - \pi_l \mathcal{P}_l}{\pi_{l+1} - \pi_l}$$

- treatment of full level 1 must be consistent with elastic top boundary condition $p_T = 0$ and mass coordinate specification $\pi_T = 0$:

$$(\mathbf{L}\mathcal{P})_1 = \frac{1}{\delta_1}\left[\left(\frac{\partial(\pi \mathcal{P})}{\partial \pi}\right)_{\tilde{1}} - \left(\frac{\partial(\pi \mathcal{P})}{\partial \pi}\right)_{\tilde{0}}\right]$$

$$\left(\frac{\partial(\pi \mathcal{P})}{\partial \pi}\right)_{\tilde{0}} = \frac{\pi_1 \mathcal{P}_1 - (\pi \mathcal{P})_{\tilde{0}}}{\pi_1 - \pi_{\tilde{0}}} = \frac{\pi_1 \mathcal{P}_1 - (p_T - \pi_T)}{\pi_1 - \pi_T} = \mathcal{P}_1$$

- treatment of full level L will be explained in separate talk devoted to bottom boundary condition

Discretization of term $\partial \mathbf{v}/\partial \pi \cdot \nabla \Psi$

- quantity $\partial \mathbf{v}/\partial \pi \cdot \nabla \Psi$ appears in X and Z terms, where Ψ is either geopotential or gw

- discretization on full levels is done using per partes rule:

$$\frac{\partial \mathbf{v}}{\partial \pi} \cdot \nabla \Psi = \frac{\partial}{\partial \pi}(\mathbf{v} \cdot \nabla \Psi) - \mathbf{v} \cdot \frac{\partial}{\partial \pi} \nabla \Psi$$

$$\left(\frac{\partial \mathbf{v}}{\partial \pi} \cdot \nabla \Psi \right)_l = \frac{\mathbf{v}_{\tilde{l}} \cdot \nabla \Psi_{\tilde{l}} - \mathbf{v}_{\tilde{l}-1} \cdot \nabla \Psi_{\tilde{l}-1}}{\delta \pi_l} - \mathbf{v}_l \cdot \frac{\nabla \Psi_{\tilde{l}} - \nabla \Psi_{\tilde{l}-1}}{\delta \pi_l} =$$

$$= \frac{(\mathbf{v}_{\tilde{l}} - \mathbf{v}_l) \cdot \nabla \Psi_{\tilde{l}} - (\mathbf{v}_l - \mathbf{v}_{\tilde{l}-1}) \cdot \nabla \Psi_{\tilde{l}-1}}{\delta \pi_l}$$

- half level velocity $\mathbf{v}_{\tilde{l}}$ is determined by interpolation ($\tilde{l} = \tilde{1}, \ldots \tilde{L} - 1$):

$$\mathbf{v}_{\tilde{l}} = \varepsilon_l \mathbf{v}_l + (1 - \varepsilon_l)\mathbf{v}_{l+1} \qquad \mathbf{v}_{\tilde{0}} = \mathbf{v}_1$$

$$\varepsilon_l = \frac{\delta_{l+1} - \alpha_{l+1}}{\delta_{l+1} - \alpha_{l+1} + \alpha_l} \qquad \mathbf{v}_{\tilde{L}} = \mathbf{v}_L$$

Discretization of term $\nabla(gw)$

- quantity $\nabla(gw)$ appears in Z term

- discretization on half level $\tilde{l}$ is straightforward:

$$gw_{\tilde{l}} = gw_{\tilde{L}} + \sum_{k=l+1}^{L} (RT)_k d_k \delta_k$$

$$\Downarrow$$

$$\nabla(gw_{\tilde{l}}) = \nabla(gw_{\tilde{L}}) + \sum_{k=l+1}^{L} \Big[\nabla(RT)_k d_k \delta_k + \\ + (RT)_k \nabla(d_k) \delta_k + (RT)_k d_k \nabla \delta_k \Big]$$

Discretization of term $\partial(\pi\mathcal{P})/\partial\pi$

- except from vertical laplacian $\mathbf{L}$, quantity $\partial(\pi\mathcal{P})/\partial\pi$ appears in pressure gradient term, where it is needed at full levels

- discretization is done in natural way:

$$\left(\frac{\partial(\pi\mathcal{P})}{\partial\pi}\right)_l = \frac{\pi_{\tilde{l}}\mathcal{P}_{\tilde{l}} - \pi_{\tilde{l}-1}\mathcal{P}_{\tilde{l}-1}}{\delta\pi_l}$$

- scaled NH pressure departure $\mathcal{P}_{\tilde{L}}$ at half levels is determined by simple averaging ($\tilde{l} = \tilde{1}, \dots, \tilde{L} - 1$):

$$\mathcal{P}_{\tilde{l}} = \frac{1}{2}(\mathcal{P}_l + \mathcal{P}_{l+1}) \qquad \begin{aligned} \mathcal{P}_{\tilde{0}} &= \mathcal{P}_1 \\ \mathcal{P}_{\tilde{L}} &= \mathcal{P}_L \end{aligned}$$

Discretization of term $\dot{\eta}\partial\Psi/\partial\eta$

- when eulerian advection is used, quantity $\dot{\eta}\partial\Psi/\partial\eta$ must be evaluated on full levels, Ψ being any prognostic variable

- discretization of vertical advection term is dictated by requirement of total energy conservation, which follows from per partes rules:

$$\int_0^1 m\dot{\eta}\frac{\partial\Psi}{\partial\eta}\,\mathrm{d}\eta = -\int_0^1 \Psi\frac{\partial}{\partial\eta}(m\dot{\eta})\,\mathrm{d}\eta$$

$$\int_0^1 m\dot{\eta}\Psi\frac{\partial\Psi}{\partial\eta}\,\mathrm{d}\eta = -\int_0^1 \frac{\Psi^2}{2}\cdot\frac{\partial}{\partial\eta}(m\dot{\eta})\,\mathrm{d}\eta$$

- in discretized case, they can be satisfied by choice:

$$\left(\dot{\eta}\frac{\partial\Psi}{\partial\eta}\right)_l = \frac{(m\dot{\eta})_{\tilde{l}}(\Psi_{l+1}-\Psi_l)+(m\dot{\eta})_{\tilde{l}-1}(\Psi_l-\Psi_{l-1})}{2\delta\pi_l}$$

- this formula is applicable also for $l=1$ and $l=L$, since $\dot{\eta}_{\tilde{0}}=\dot{\eta}_{\tilde{L}}=0$

Discretization of term $\nabla\pi/\pi$ (1)

- conservation of net angular momentum in global atmosphere is a consequence of continuous identity:

$$\int_0^1 \frac{mRT}{p}\nabla p \,\mathrm{d}\eta = \int_0^1 (\phi - \phi_S)\nabla\frac{\partial p}{\partial\eta}\,\mathrm{d}\eta$$

- its discrete counterpart can be satisfied by following choice:

$$\left(\frac{\nabla p}{p}\right)_l = \frac{1}{1+\mathcal{P}_l}\cdot\frac{1}{\delta\pi_l}\left[\alpha_l\nabla(\delta p_l) + \delta_l\sum_{k=1}^{l-1}\nabla(\delta p_k)\right]$$

- application of this formula in pressure gradient term led to unstable behaviour due to incompatible $\nabla\mathcal{P}$ discretization in linear SI system and non-linear model

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- application of this formula in pressure gradient term led to unstable behaviour due to incompatible $\nabla\mathcal{P}$ discretization in linear SI system and non-linear model $\Rightarrow$ requirement of net angular momentum conservation is in conflict with model stability

Discretization of term $\nabla\pi/\pi$ (2)

- situation can be solved by splitting term $\nabla p/p$ into hydrostatic and non-hydrostatic parts:

$$\frac{\nabla p}{p} = \frac{\nabla[\pi(1+\mathcal{P})]}{\pi(1+\mathcal{P})} = \frac{\nabla\pi}{\pi} + \frac{\nabla\mathcal{P}}{1+\mathcal{P}}$$

- angular momentum conservation is imposed only on hydrostatic part:

$$\left(\frac{\nabla\pi}{\pi}\right)_l = \frac{1}{\delta\pi_l}\left[\alpha_l\nabla(\delta\pi_l) + \delta_l\sum_{k=1}^{l-1}\nabla(\delta\pi_k)\right] =$$

$$= \frac{1}{\delta\pi_l}\left[\alpha_l\nabla(\pi_{\tilde{l}} - \pi_{\tilde{l}-1}) + \delta_l\nabla\pi_{\tilde{l}-1}\right] = \frac{1}{\delta\pi_l}\left[\frac{C_l}{\pi_l} + \delta B_l\right]\nabla\pi_{\tilde{L}}$$

$$C_l = A_{\tilde{l}}B_{\tilde{l}-1} - A_{\tilde{l}-1}B_{\tilde{l}} \qquad \delta B_l = B_{\tilde{l}} - B_{\tilde{l}-1}$$

- term $\nabla\mathcal{P}/(1+\mathcal{P})$ is discretized in natural way

Discretization of term $\nabla\phi$

- quantity $\nabla\phi$ occurs in pressure gradient term and in X term

- it is needed on both half and full levels:

$$\phi_{\tilde{l}} = \phi_{\tilde{L}} + \sum_{k=l+1}^{L} \frac{(RT)_k}{1+\mathcal{P}_k}\delta_k \qquad \phi_l = \phi_{\tilde{l}} + \frac{(RT)_l}{1+\mathcal{P}_l}\alpha_l$$

$$\Downarrow$$

$$\nabla\phi_{\tilde{l}} = \nabla\phi_{\tilde{L}} + \sum_{k=l+1}^{L}\left[\frac{\nabla(RT)_k}{1+\mathcal{P}_k}\delta_k - \frac{(RT)_k\nabla\mathcal{P}_k}{(1+\mathcal{P}_k)^2}\delta_k + \frac{(RT)_k}{1+\mathcal{P}_k}\nabla\delta_k\right]$$

$$\nabla\phi_l = \nabla\phi_{\tilde{l}} + \frac{\nabla(RT)_l}{1+\mathcal{P}_l}\alpha_l - \frac{(RT)_l\nabla\mathcal{P}_l}{(1+\mathcal{P}_l)^2}\alpha_l + \frac{(RT)_l}{1+\mathcal{P}_l}\nabla\alpha_l$$

Discretization of terms $\nabla\alpha_l$, $\nabla\delta_l$

- discretization of quantities $\nabla\alpha_l$, $\nabla\delta_l$ follows directly from their definitions ($l = 2, \dots, L$):

$$\nabla\alpha_l = -\frac{1}{2} \cdot \frac{C_l}{\pi_{\tilde{l}}\pi_{\tilde{l}-1}}(1-\alpha_l)\nabla\pi_{\tilde{L}} \qquad \nabla\alpha_1 = 0$$

$$\nabla\delta_l = -\frac{1}{2} \cdot \frac{C_l}{\pi_{\tilde{l}}\pi_{\tilde{l}-1}} \cdot \frac{1+(1-\alpha_l)^2}{1-\alpha_l}\nabla\pi_{\tilde{L}} \qquad \nabla\delta_1 = 0$$

$$C_l = A_{\tilde{l}}B_{\tilde{l}-1} - A_{\tilde{l}-1}B_{\tilde{l}}$$

Conclusions

- design of reliable vertical discretization is a difficult task
- only restricted set of continuous properties can be preserved by discretized scheme
- this set must be selected very carefully, since the wrong choices easily lead to model instability
- another source of problems can be inconsistency between linear SI system and non-linear model
- sometimes conservation constraints must be relaxed in favour of model stability