Continuity equation for non-spherical geometries in mass-coordinates

Pierre Bénard
CNRM/GMAP/ALGO 03 June 2014
file : Continuity_EGA.tex

1 Introduction

The main goal of this paper is to express the continuity equation for a quite general non-spherical geometry, but in mass coordinate. The framework is similar to the one of White and Wood, 2012 (QJRMS, **138**, 980–988, WW12 hereafter), but here, the continuity equation is expressed in mass-coordinates in order to make apparent the way by which the continuity equation of ARPEGE should be generalized to a non-spherical geometry in various cases. References are also made to Bénard, 2014, (QJRMS, **140**, 170–184, B14a hereafter) and Bénard, 2014 (QJRMS, in Press, 2014b herafter).

2 Notations and framework

As in WW12, a general curvilinear coordinate system attached to the rotating Earth (ξ_1,ξ_2,ξ_3) is considered. Two main properties are assumed: (i) the coordinate system is orthogonal and axially symmetric; (ii) the coordinate system is "horizontal/vertical" (in B14a's terminology) or a "geopotential" coordinate system (in WW12's terminology).

2.1 Orthogonal axially-symmetric curvilinear coordinate systems

The orthonormal basis along increasing coordinate directions of (ξ_1,ξ_2,ξ_3) respectively is noted $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. Hence we have

$$\mathbf{e}_i.\mathbf{e}_j = \delta_{ij}, \tag{1}$$

where δ_{ij} is the Kronecker symbol.

The metrics of this coordinate system is defined through the metric factors (h_1,h_2,h_3) which may be expressed by:

$$h_i^2 = \left(\frac{\partial x}{\partial \xi_i}\right)^2 + \left(\frac{\partial y}{\partial \xi_i}\right)^2 + \left(\frac{\partial z}{\partial \xi_i}\right)^2 \quad \text{for } i=1,2,3, \tag{2}$$

where (x,y,z) is a standard Cartesian system also attached to the rotating Earth framework. The distance element is therefore

$$\mathrm{d}s^2 = h_1^2\,\mathrm{d}\xi_1^2 + h_2^2\,\mathrm{d}\xi_2^2 + h_3^2\,\mathrm{d}\xi_3^2. \tag{3}$$

And the determinant of the metric is denoted by $h = h_1h_2h_3$. The axial symmetry (independance of all metric factors to the longitude) is expressed by:

$$\frac{\partial h_1}{\partial \xi_1} = \frac{\partial h_2}{\partial \xi_1} = \frac{\partial h_3}{\partial \xi_1} = \frac{\partial h}{\partial \xi_1} = 0. \tag{4}$$

The wind vector $\mathbf{V}$ is decomposed in the basis as $\mathbf{V} = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2 + u_3\,\mathbf{e}_3$, and by definition, we have

$$\begin{aligned} u_1 &= h_1\,\dot{\xi}_1, \\ u_2 &= h_2\,\dot{\xi}_2, \\ u_3 &= h_3\,\dot{\xi}_3. \end{aligned} \tag{5}$$

Since the coordinate system is orthogonal, the vector calculus operators have simple expressions, without any non-diagonal metric term. The gradient of a scalar field ψ is

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial\xi_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial\xi_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial\xi_3}\mathbf{e}_3. \tag{6}$$

The divergence of a vector field given by $\mathbf{A} = A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3$ is

$$\nabla.\mathbf{A} = \frac{1}{h}\left[\frac{\partial}{\partial\xi_1}\left(\frac{A_1 h}{h_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{A_2 h}{h_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{A_3 h}{h_3}\right)\right]. \tag{7}$$

2.2 Horizontal/vertical coordinate systems

The coordinate system is assumed to be "horizontal/vertical", which means that surfaces of constant values ξ_1 and ξ_2 are vertical, and surfaces of constant ξ_3 values are horizontal. Moreover, ξ_1 points eastward, ξ_2 points northward, and ξ_3 upward. This system is not assumed to be limited to the polar spherical coordinates, but is more general. For instance if the shape of horizontal surfaces is non-spherical, and the vertical lines are curved, the coordinate system is not the spherical coordinate system. In spherical geometry, ξ_1,ξ_2 and ξ_3 could be defined as longitude, latitude and geocentric radius respectively. In a non-spherical geometry (i.e. horizontal surfaces are not spheres), due to the axial symmetry, ξ_1 may still be viewed and called as "longitude", but ξ_2 and ξ_3 have no standard names or representation. An horizontal surface is, by definition, an iso-geopotential surface: $\phi = \mathrm{const.}$. Hence, the fact that iso-ξ_3 surfaces are also horizontal is expressed as:

$$\frac{\partial\phi}{\partial\xi_1} = \frac{\partial\phi}{\partial\xi_2} = 0 \tag{8}$$

which may also be expressed as

$$\frac{\partial\phi}{\partial\xi_3} = \frac{\mathrm{d}\phi}{\mathrm{d}\xi_3} \tag{9}$$

or

$$\phi = \phi(\xi_3) \tag{10}$$

N.B.: In all this paper, we adopt a "meteorological" convention for the geopotential: the geopotential is assumed to *increase* with geocentric radius. This contasts with "astronomic" contexts, where the geopotential is commonly assumed to decrease with geocentric radius (in order to asymptotically reach zero at infinite). Hence we define the "geopotential" ϕ here as the <u>opposite</u> of the scalar potential field from with the vector gravity field is derived. From (6), (8) and (9), we therefore have:

$$\mathbf{g} = -\nabla.\phi = -\frac{1}{h_3}\frac{\mathrm{d}\phi}{\mathrm{d}\xi_3}\mathbf{e}_3. \tag{11}$$

Projecting this equation along $\mathbf{e}_3$ gives the following relationship between the geopotential and the intensity of gravity g:

$$g = \frac{1}{h_3}\frac{\mathrm{d}\phi}{\mathrm{d}\xi_3}, \tag{12}$$

where all quantities g, h_3 and $(\mathrm{d}\phi/\mathrm{d}\xi_3)$ are positive.
The product gh_3 is given by

$$gh_3 = \frac{\mathrm{d}\phi}{\mathrm{d}\xi_3}. \tag{13}$$

Consequently, in the most general non-spherical deep-atmosphere framework, all metric factors exhibit horizontal and vertical variations as well as the apparent gravity g, but the product gh_3 is only vertically varying

$$\frac{\partial(gh_3)}{\partial\xi_1} = \frac{\partial(gh_3)}{\partial\xi_2} = 0. \tag{14}$$

However, if ξ_3 is chosen as being proportional to ϕ, this product gh_3 is a constant in space and time. If, in particular, we choose exactly $\xi_3 = \phi$, then gh_3 is constant in space and time and equal to one, as indicated by (13).

3 Mass-based coordinates: transformation formulae

A new space-time coordinate system (ξ_1', ξ_2', s, t') is assumed in addition to (ξ_1, ξ_2, ξ_3, t). The only difference between the two systems is in the vertical coordinate hence we have

$$\begin{aligned} \xi_1'(\xi_1, \xi_2, \xi_3, t) &= \xi_1, \\ \xi_2'(\xi_1, \xi_2, \xi_3, t) &= \xi_2, \\ t'(\xi_1, \xi_2, \xi_3, t) &= t. \end{aligned}$$

In terms of derivatives, we therefore have:

$$\begin{aligned} (\partial/\partial s)|_{(\xi_1,\xi_2,t)} &= (\partial/\partial s)|_{(\xi_1',\xi_2',t')} \\ (\partial/\partial \xi_3)|_{(\xi_1,\xi_2,t)} &= (\partial/\partial \xi_3)|_{(\xi_1',\xi_2',t')}, \end{aligned}$$

where the subscripts indicate what is held constant. The time and horizontal coordinates do not need to be distinguished any longer between the two systems, and the primes are therefore dropped from now on. In a partial derivative symbol, the only possible ambiguity about what is held constant involves s or ξ_3 in horizontal or time derivatives, hence in the following, the subscript notation is kept only for these two coordinates. The method to introduce a new time-dependent vertical coordinate is not original, and exactly follows Kasahara (1974), except that the framework is not Cartesian.

For any scalar ψ, the total differential is given by

$$\mathrm{d}\psi = \left(\frac{\partial\psi}{\partial t}\right)_{\xi_3} \mathrm{d}t + \left(\frac{\partial\psi}{\partial\xi_1}\right)_{\xi_3} \mathrm{d}\xi_1 + \left(\frac{\partial\psi}{\partial\xi_2}\right)_{\xi_3} \mathrm{d}\xi_2 + \left(\frac{\partial\psi}{\partial\xi_3}\right) \mathrm{d}\xi_3 \tag{15}$$

$$\mathrm{d}\psi = \left(\frac{\partial\psi}{\partial t}\right)_{s} \mathrm{d}t + \left(\frac{\partial\psi}{\partial\xi_1}\right)_{s} \mathrm{d}\xi_1 + \left(\frac{\partial\psi}{\partial\xi_2}\right)_{s} \mathrm{d}\xi_2 + \left(\frac{\partial\psi}{\partial s}\right) \mathrm{d}s \tag{16}$$

Applying the two previous equations to s and ξ_3 respectively yields

$$\mathrm{d}s = \left(\frac{\partial s}{\partial t}\right)_{\xi_3} \mathrm{d}t + \left(\frac{\partial s}{\partial\xi_1}\right)_{\xi_3} \mathrm{d}\xi_1 + \left(\frac{\partial s}{\partial\xi_2}\right)_{\xi_3} \mathrm{d}\xi_2 + \left(\frac{\partial s}{\partial\xi_3}\right) \mathrm{d}\xi_3 \tag{17}$$

$$\mathrm{d}\xi_3 = \left(\frac{\partial\xi_3}{\partial t}\right)_{s} \mathrm{d}t + \left(\frac{\partial\xi_3}{\partial\xi_1}\right)_{s} \mathrm{d}\xi_1 + \left(\frac{\partial\xi_3}{\partial\xi_2}\right)_{s} \mathrm{d}\xi_2 + \left(\frac{\partial\xi_3}{\partial s}\right) \mathrm{d}s \tag{18}$$

From (15) and (18):

$$
\begin{aligned}
\mathrm{d}\psi = & \left(\frac{\partial\psi}{\partial t}\right)_{\xi_3} \mathrm{d}t + \left(\frac{\partial\psi}{\partial\xi_1}\right)_{\xi_3} \mathrm{d}\xi_1 + \left(\frac{\partial\psi}{\partial\xi_2}\right)_{\xi_3} \mathrm{d}\xi_2 \\
& + \left(\frac{\partial\psi}{\partial\xi_3}\right) \left[\left(\frac{\partial\xi_3}{\partial t}\right)_s \mathrm{d}t + \left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s \mathrm{d}\xi_1 + \left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s \mathrm{d}\xi_2 + \left(\frac{\partial\xi_3}{\partial s}\right) \mathrm{d}s \right] \\
= & \left[\left(\frac{\partial\psi}{\partial t}\right)_{\xi_3} + \left(\frac{\partial\psi}{\partial\xi_3}\right) \left(\frac{\partial\xi_3}{\partial t}\right)_s \right] \mathrm{d}t + \left[\left(\frac{\partial\psi}{\partial\xi_1}\right)_{\xi_3} + \left(\frac{\partial\psi}{\partial\xi_3}\right) \left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s \right] \mathrm{d}\xi_1 \\
& + \left[\left(\frac{\partial\psi}{\partial\xi_2}\right)_{\xi_3} + \left(\frac{\partial\psi}{\partial\xi_3}\right) \left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s \right] \mathrm{d}\xi_2 + \left(\frac{\partial\psi}{\partial\xi_3}\right) \left(\frac{\partial\xi_3}{\partial s}\right) \mathrm{d}s
\end{aligned} \tag{19}
$$

Similarly, from (16) and (17):

$$
\begin{aligned}
\mathrm{d}\psi = & \left(\frac{\partial\psi}{\partial t}\right)_s \mathrm{d}t + \left(\frac{\partial\psi}{\partial\xi_1}\right)_s \mathrm{d}\xi_1 + \left(\frac{\partial\psi}{\partial\xi_2}\right)_s \mathrm{d}\xi_2 \\
& + \left(\frac{\partial\psi}{\partial s}\right) \left[\left(\frac{\partial s}{\partial t}\right)_{\xi_3} \mathrm{d}t + \left(\frac{\partial s}{\partial\xi_1}\right)_{\xi_3} \mathrm{d}\xi_1 + \left(\frac{\partial s}{\partial\xi_2}\right)_{\xi_3} \mathrm{d}\xi_2 + \left(\frac{\partial s}{\partial\xi_3}\right) \mathrm{d}\xi_3 \right] \\
= & \left[\left(\frac{\partial\psi}{\partial t}\right)_s + \left(\frac{\partial\psi}{\partial s}\right) \left(\frac{\partial s}{\partial t}\right)_{\xi_3} \right] \mathrm{d}t + \left[\left(\frac{\partial\psi}{\partial\xi_1}\right)_s + \left(\frac{\partial\psi}{\partial s}\right) \left(\frac{\partial s}{\partial\xi_1}\right)_{\xi_3} \right] \mathrm{d}\xi_1 \\
& + \left[\left(\frac{\partial\psi}{\partial\xi_2}\right)_s + \left(\frac{\partial\psi}{\partial s}\right) \left(\frac{\partial s}{\partial\xi_2}\right)_{\xi_3} \right] \mathrm{d}\xi_2 + \left(\frac{\partial\psi}{\partial s}\right) \left(\frac{\partial s}{\partial\xi_3}\right) \mathrm{d}\xi_3
\end{aligned} \tag{20}
$$

Identifying the factors of dt, $d\xi_1$, $d\xi_2$, $d\xi_3$ and ds in (15), (20) and (16), (19) provides the coordinate transformation formulae:

$$\left(\frac{\partial\psi}{\partial c}\right)_s = \left(\frac{\partial\psi}{\partial c}\right)_{\xi_3} + \frac{\partial\psi}{\partial\xi_3} \left(\frac{\partial\xi_3}{\partial c}\right)_s \tag{21}$$

$$\left(\frac{\partial\psi}{\partial c}\right)_{\xi_3} = \left(\frac{\partial\psi}{\partial c}\right)_s + \frac{\partial\psi}{\partial s} \left(\frac{\partial s}{\partial c}\right)_{\xi_3} \tag{22}$$

$$\left(\frac{\partial\psi}{\partial\xi_3}\right) = \left(\frac{\partial\psi}{\partial s}\right) \left(\frac{\partial s}{\partial\xi_3}\right) \tag{23}$$

$$\left(\frac{\partial\psi}{\partial s}\right) = \left(\frac{\partial\psi}{\partial\xi_3}\right) \left(\frac{\partial\xi_3}{\partial s}\right) \tag{24}$$

where c stands for t, ξ_1 or ξ_2. Applying the second to ξ_3 and the first to s, we have

$$\left(\frac{\partial\xi_3}{\partial c}\right)_s = -\frac{\partial\xi_3}{\partial s} \left(\frac{\partial s}{\partial c}\right)_{\xi_3} \tag{25}$$

$$\left(\frac{\partial s}{\partial c}\right)_{\xi_3} = -\frac{\partial s}{\partial\xi_3} \left(\frac{\partial\xi_3}{\partial c}\right)_s \tag{26}$$

From (15), (16) the total time-derivative writes, in the two systems

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \left(\frac{\partial\psi}{\partial t}\right)_{\xi_3} + \left(\frac{\partial\psi}{\partial\xi_1}\right)_{\xi_3} \dot{\xi}_1 + \left(\frac{\partial\psi}{\partial\xi_2}\right)_{\xi_3} \dot{\xi}_2 + \left(\frac{\partial\psi}{\partial\xi_3}\right) \dot{\xi}_3 \tag{27}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \left(\frac{\partial\psi}{\partial t}\right)_s + \left(\frac{\partial\psi}{\partial\xi_1}\right)_s \dot{\xi}_1 + \left(\frac{\partial\psi}{\partial\xi_2}\right)_s \dot{\xi}_2 + \left(\frac{\partial\psi}{\partial s}\right) \dot{s} \tag{28}$$

4 Continuity equation in mass-based coordinates (partial form)

A first form of the continuity equation in s coordinate is now derived. This first form, simpler than the second form derived in next section, is only partial in that it may be used in practice only in the case where ξ_3 is the true geopotential itself.

The continuity equation is, in space-coordinate independent form

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\nabla.\mathbf{V}=0. \tag{29}$$

In ξ_3 coordinate, the wind divergence writes, using (5) and (7)

$$\nabla.\mathbf{V}=\frac{1}{h}\left[\frac{\partial}{\partial\xi_1}\left(h\dot{\xi}_1\right)_{\xi_3}+\frac{\partial}{\partial\xi_2}\left(h\dot{\xi}_2\right)_{\xi_3}+\frac{\partial}{\partial\xi_3}\left(h\dot{\xi}_3\right)\right],$$

i.e., taking into account the axial symmetry

$$\nabla.\mathbf{V}=\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_{\xi_3}+\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_{\xi_3}+\left(\frac{\partial\dot{\xi}_3}{\partial\xi_3}\right)+\frac{1}{h}\left[\dot{\xi}_2\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}+\dot{\xi}_3\left(\frac{\partial h}{\partial\xi_3}\right)\right] \tag{30}$$

The term $(\partial\dot{\xi}_3/\partial\xi_3)$ may be expressed in the new coordinate using the transformations rule (23) and (28) applied to ξ_3:

$$\begin{aligned}\left(\frac{\partial\dot{\xi}_3}{\partial\xi_3}\right)&=\frac{\partial s}{\partial\xi_3}\frac{\partial\dot{\xi}_3}{\partial s}\\&=\frac{\partial s}{\partial\xi_3}\left[\frac{\partial}{\partial s}\left(\frac{\partial\xi_3}{\partial t}\right)_s+\dot{\xi}_1\frac{\partial}{\partial s}\left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s+\left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s\frac{\partial\dot{\xi}_1}{\partial s}\right.\\&\left.\quad+\dot{\xi}_2\frac{\partial}{\partial s}\left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s+\left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s\frac{\partial\dot{\xi}_2}{\partial s}+\dot{s}\frac{\partial}{\partial s}\left(\frac{\partial\xi_3}{\partial s}\right)+\left(\frac{\partial\xi_3}{\partial s}\right)\frac{\partial\dot{s}}{\partial s}\right]\end{aligned}$$

Finally:

$$\left(\frac{\partial\dot{\xi}_3}{\partial\xi_3}\right)=\frac{\partial s}{\partial\xi_3}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_3}{\partial s}\right)+\left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s\frac{\partial\dot{\xi}_1}{\partial s}+\left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s\frac{\partial\dot{\xi}_2}{\partial s}\right]+\frac{\partial\dot{s}}{\partial s} \tag{31}$$

From (22) applied to $\dot{\xi}_1$, $\dot{\xi}_2$ and using (25), the two first terms of the divergence in (30) may be expressed by

$$\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_{\xi_3}=\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_s-\frac{\partial\dot{\xi}_1}{\partial s}\frac{\partial s}{\partial\xi_3}\left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s \tag{32}$$

$$\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_{\xi_3}=\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_s-\frac{\partial\dot{\xi}_2}{\partial s}\frac{\partial s}{\partial\xi_3}\left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s. \tag{33}$$

The continuity equation may then be derived in s coordinate, as shown below, starting from (29), (30):

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_{\xi_3}+\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_{\xi_3}+\left(\frac{\partial\dot{\xi}_3}{\partial\xi_3}\right)+\frac{1}{h}\left[\dot{\xi}_2\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}+\dot{\xi}_3\left(\frac{\partial h}{\partial\xi_3}\right)\right]=0.$$

$\Longrightarrow$

$$
\begin{aligned}
&\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_s-\frac{\partial\dot{\xi}_1}{\partial s}\frac{\partial s}{\partial\xi_3}\left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s+\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_s-\frac{\partial\dot{\xi}_2}{\partial s}\frac{\partial s}{\partial\xi_3}\left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s+\frac{\dot{\xi}_2}{h}\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}\\
&\quad+\frac{\partial s}{\partial\xi_3}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_3}{\partial s}\right)+\left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s\frac{\partial\dot{\xi}_1}{\partial s}+\left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s\frac{\partial\dot{\xi}_2}{\partial s}\right]+\frac{\partial\dot{s}}{\partial s}+\frac{\dot{\xi}_3}{h}\left(\frac{\partial h}{\partial\xi_3}\right)=0.
\end{aligned}
$$

After cancellation of four term:

$$
\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\frac{\partial s}{\partial\xi_3}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_3}{\partial s}\right)\right]+\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_s+\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_s+\frac{\dot{\xi}_2}{h}\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}+\frac{\partial\dot{s}}{\partial s}+\frac{\dot{\xi}_3}{h}\left(\frac{\partial h}{\partial\xi_3}\right)=0.
$$

$\Longrightarrow$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\left(\rho\frac{\partial\xi_3}{\partial s}\right)\right]+\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_s+\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_s+\frac{\partial\dot{s}}{\partial s}+\frac{\dot{\xi}_2}{h}\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}+\frac{\dot{\xi}_3}{h}\left(\frac{\partial h}{\partial\xi_3}\right)=0. \tag{34}
$$

The total derivative may now be expanded in s coordinate

$$
\begin{aligned}
&\frac{1}{\rho}\frac{\partial s}{\partial\xi_3}\left[\frac{\partial}{\partial t}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s+\dot{\xi}_1\frac{\partial}{\partial\xi_1}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s+\dot{\xi}_2\frac{\partial}{\partial\xi_2}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s+\dot{s}\frac{\partial}{\partial s}\left(\rho\frac{\partial\xi_3}{\partial s}\right)\right]\\
&\quad+\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_s+\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_s+\frac{\partial\dot{s}}{\partial s}+\frac{\dot{\xi}_2}{h}\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}+\frac{\dot{\xi}_3}{h}\left(\frac{\partial h}{\partial\xi_3}\right)=0.
\end{aligned} \tag{35}
$$

$\Longrightarrow$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s+\dot{\xi}_1\frac{\partial}{\partial\xi_1}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s+\rho\frac{\partial\xi_3}{\partial s}\left(\frac{\partial\dot{\xi}_1}{\partial\xi_1}\right)_s+\dot{\xi}_2\frac{\partial}{\partial\xi_2}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s+\rho\frac{\partial\xi_3}{\partial s}\left(\frac{\partial\dot{\xi}_2}{\partial\xi_2}\right)_s\\
&+\dot{s}\frac{\partial}{\partial s}\left(\rho\frac{\partial\xi_3}{\partial s}\right)+\rho\frac{\partial\xi_3}{\partial s}\left(\frac{\partial\dot{s}}{\partial s}\right)+\rho\frac{\partial\xi_3}{\partial s}\left[\frac{\dot{\xi}_2}{h}\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}+\frac{\dot{\xi}_3}{h}\left(\frac{\partial h}{\partial\xi_3}\right)\right]=0.
\end{aligned} \tag{36}
$$

$\Longrightarrow$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s+\frac{\partial}{\partial\xi_1}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{\xi}_1\right)_s+\frac{\partial}{\partial\xi_2}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{\xi}_2\right)_s+\frac{\partial}{\partial s}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{s}\right)\\
&+\rho\frac{\partial\xi_3}{\partial s}\left[\frac{\dot{\xi}_2}{h}\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}+\frac{\dot{\xi}_3}{h}\left(\frac{\partial h}{\partial\xi_3}\right)\right]=0.
\end{aligned}
$$

The last two terms may be expressed in invariant form since

$$
\left[\frac{\dot{\xi}_2}{h}\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3}+\frac{\dot{\xi}_3}{h}\left(\frac{\partial h}{\partial\xi_3}\right)\right]=\frac{1}{h}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)=\frac{\dot{h}}{h} \tag{37}
$$

We finally obtain a first form of the continuity equation is s coordinate

$$
\frac{\partial}{\partial t}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s+\frac{\partial}{\partial\xi_1}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{\xi}_1\right)_s+\frac{\partial}{\partial\xi_2}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{\xi}_2\right)_s+\frac{\partial}{\partial s}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{s}\right)+\rho\frac{\partial\xi_3}{\partial s}\frac{\dot{h}}{h}=0. \tag{38}
$$

Setting $\xi_3=\phi$, the continuity equation becomes:

$$
\frac{\partial}{\partial t}\left(\rho\frac{\partial\phi}{\partial s}\right)_s+\frac{\partial}{\partial\xi_1}\left(\rho\frac{\partial\phi}{\partial s}\dot{\xi}_1\right)_s+\frac{\partial}{\partial\xi_2}\left(\rho\frac{\partial\phi}{\partial s}\dot{\xi}_2\right)_s+\frac{\partial}{\partial s}\left(\rho\frac{\partial\phi}{\partial s}\dot{s}\right)+\rho\frac{\partial\phi}{\partial s}\frac{\dot{h}}{h}=0. \tag{39}
$$

5 Continuity equation in the new vertical coordinate (complete form)

N.B.: In this section, we want to derive a more general form than (38), in which the evolutive quantity is $gh_3\rho(\partial\xi_3/\partial s)$. The purpose of this section is only for completeness of algebraic derivations. All results presented in this section have no practical consequences for the remainder of the paper, because all practical results presented in subsequent sections may be derived from (39). This section may therefore be skipped without any problem.

We have

$$\frac{\partial}{\partial t}\left(gh_3\rho\frac{\partial\xi_3}{\partial s}\right)_s = gh_3\frac{\partial}{\partial t}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s + \left(\rho\frac{\partial\xi_3}{\partial s}\right)\left(\frac{\partial gh_3}{\partial t}\right)_s \tag{40}$$

The first rhs term may be readily expressed from (38). The time derivative in the second rhs term may be expressed by

$$\begin{aligned}\left(\frac{\partial gh_3}{\partial t}\right)_s &= \left(\frac{\partial gh_3}{\partial t}\right)_{\xi_3} + \frac{\mathrm{d}\,gh_3}{\mathrm{d}\xi_3}\left(\frac{\partial\xi_3}{\partial t}\right)_s \\ &= \frac{\mathrm{d}gh_3}{\mathrm{d}\xi_3}\left[\dot\xi_3 - \dot s\left(\frac{\partial\xi_3}{\partial s}\right) - \dot\xi_1\left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s - \dot\xi_2\left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s\right],\end{aligned} \tag{41}$$

where (28) has been used to express $(\partial\xi_3/\partial t)_s$. Hence we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(gh_3\rho\frac{\partial\xi_3}{\partial s}\right)_s = &-gh_3\left[+\frac{\partial}{\partial\xi_1}\left(\rho\frac{\partial\xi_3}{\partial s}\dot\xi_1\right)_s + \frac{\partial}{\partial\xi_2}\left(\rho\frac{\partial\xi_3}{\partial s}\dot\xi_2\right)_s + \frac{\partial}{\partial s}\left(\rho\frac{\partial\xi_3}{\partial s}\dot s\right) + \rho\frac{\partial\xi_3}{\partial s}\frac{\dot h}{h}\right] \\ &+\left(\rho\frac{\partial\xi_3}{\partial s}\right)\frac{\mathrm{d}\,gh_3}{\mathrm{d}\xi_3}\left[\dot\xi_3 - \dot s\left(\frac{\partial\xi_3}{\partial s}\right) - \dot\xi_1\left(\frac{\partial\xi_3}{\partial\xi_1}\right)_s - \dot\xi_2\left(\frac{\partial\xi_3}{\partial\xi_2}\right)_s\right].\end{aligned} \tag{42}$$

Grouping terms involving $\dot\xi_1$, $\dot\xi_2$ and $\dot s$ leads, after some simple algebra, to

$$\frac{\partial}{\partial t}\left(gh_3\rho\frac{\partial\xi_3}{\partial s}\right)_s + \frac{\partial}{\partial\xi_1}\left(gh_3\rho\frac{\partial\xi_3}{\partial s}\dot\xi_1\right)_s + \frac{\partial}{\partial\xi_2}\left(gh_3\rho\frac{\partial\xi_3}{\partial s}\dot\xi_2\right)_s + \frac{\partial}{\partial s}\left(gh_3\rho\frac{\partial\xi_3}{\partial s}\dot s\right) + gh_3\rho\frac{\partial\xi_3}{\partial s}\left[\frac{\dot h}{h} - \frac{\dot{\overline{gh_3}}}{gh_3}\right] = 0. \tag{43}$$

where it has also been made use of the fact that $gh_3 = gh_3(\xi_3)$ and $h = h(\xi_2,\xi_3)$, which implies

$$\dot h = \dot\xi_2\left(\frac{\partial h}{\partial\xi_2}\right)_{\xi_3} + \dot\xi_3\frac{\partial h}{\partial\xi_3} \tag{44}$$

$$\dot{\overline{gh_3}} = \dot\xi_3\frac{\mathrm{d}\,gh_3}{\mathrm{d}\xi_3}. \tag{45}$$

It is seen that in the particular case $\xi_3 = \phi$, where $gh_3 = 1$, the complete and partial forms are identical.

Since $h/(gh_3) = h_1h_2/g$, the last term inside brackets in (43) may be rewritten in a form which do not depend any longer on the original vertical coordinate:

$$\left[\frac{\dot h}{h} - \frac{\dot{\overline{gh_3}}}{gh_3}\right] = \dot{\overline{\left(\frac{h_1h_2}{g}\right)}}\frac{g}{h_1h_2} \tag{46}$$

Another equivalent form of the continuity equation is therefore

$$\frac{\partial}{\partial t}\left(gh_3\rho\frac{\partial \xi_3}{\partial s}\right)_s+\frac{\partial}{\partial \xi_1}\left(gh_3\rho\frac{\partial \xi_3}{\partial s}\dot{\xi}_1\right)_s+\frac{\partial}{\partial \xi_2}\left(gh_3\rho\frac{\partial \xi_3}{\partial s}\dot{\xi}_2\right)_s+\frac{\partial}{\partial s}\left(gh_3\rho\frac{\partial \xi_3}{\partial s}\dot{s}\right)+gh_3\rho\frac{\partial \xi_3}{\partial s}\left[\overset{\cdot}{\overline{\left(\frac{h_1h_2}{g}\right)}}\frac{g}{h_1h_2}\right]=0. \tag{47}$$

6 Mass-based coordinates for deep-atmosphere systems

6.1 Hydrostatic-pressure coordinate π

The true hydrostatic pressure π may be deduced from (A.15) of WW12, as the pressure in the quasi-hydrostatic case when all winds vanish, which therefore yields

$$\left(\frac{\partial \pi}{\partial \xi_3}\right)=-\rho g h_3. \tag{48}$$

In non-hydrostatic systems, p is not equal to π. We see that additionally, in the deep-atmosphere quasi-hydrostatic system the quasi-hydrostatic pressure p is also not equal to the hydrostatic pressure π, due to terms involving the horizontal wind in WW12's (A.15).

The fact that ρ, g and h_3 do not vanish in the domain ensures that π may be chosen as a vertical coordinate. Doing so, the continuity equation in π coordinate may be derived by simply substituting $s\to\pi$ in (47). Then, we have

$$\rho g h_3\left(\frac{\partial \xi_3}{\partial \pi}\right)=-1. \tag{49}$$

and the continuity equation becomes

$$\left(\frac{\partial \dot{\xi}_1}{\partial \xi_1}\right)_\pi+\left(\frac{\partial \dot{\xi}_2}{\partial \xi_2}\right)_\pi+\left(\frac{\partial \dot{\pi}}{\partial \pi}\right)+\left[\overset{\cdot}{\overline{\left(\frac{h_1h_2}{g}\right)}}\frac{g}{h_1h_2}\right]=0. \tag{50}$$

The way the last term may be diagnosed is not examined further since π is not a coordinate that is used in practice, because the fixed-in-space bottom boundary is not easy to represent in this coordinate.
The continuity equation (50) might also have been directly derived from (38), with the additional particular choice $\xi_3=\phi$, for which $h_3=1/g$, hence $h=h_1h_2/g$.

6.2 Terrain-following Hydrostatic-pressure coordinate σ

We set $s=\sigma=\pi/\pi_s$, where π_s is the hydrostatic surface pressure (i.e. the value of π at the rigid bottom surface). Hence we have

$$\left(\frac{\partial s}{\partial \xi_3}\right)=\left(\frac{\partial \sigma}{\partial \xi_3}\right)=\frac{1}{\pi_s}\frac{\partial \pi}{\partial \xi_3}=-\frac{\rho g h_3}{\pi_s}. \tag{51}$$

The continuity equation in σ coordinate then writes

$$\frac{\partial \pi_s}{\partial t}+\frac{\partial}{\partial \xi_1}\left(\pi_s\dot{\xi}_1\right)_\sigma+\frac{\partial}{\partial \xi_2}\left(\pi_s\dot{\xi}_2\right)_\sigma+\frac{\partial \dot{\sigma}}{\partial \sigma}+\pi_s\left[\overset{\cdot}{\overline{\left(\frac{h_1h_2}{g}\right)}}\frac{g}{h_1h_2}\right]=0. \tag{52}$$

The vertical integration of the geopotential depth element gives

$$\phi=\phi_s+\pi_s\int_\sigma^1\frac{1}{\rho}\mathrm{d}\sigma=\phi_s+\pi_s\int_\sigma^1\frac{RT}{p}\mathrm{d}\sigma. \tag{53}$$

It is noteworthy that in the deep-atmosphere quasi-hydrostatic context, ρRT equals p, not π, and $p \neq \pi$. Hence the problem of the formulation of a quasi-hydrostatic deep-atmosphere model in mass-based terrain-following coordinate is not as simple as in the shallow-atmosphere framework, examined in next section.

6.3 Hybrid hydrostatic-pressure terrain-following coordinate η

This coordinate $s = \eta$ is implicitly defined by $\pi = A(\eta) + \pi_s B(\eta)$ which implies the following definition and expression for m :

$$m := \frac{\partial \pi}{\partial \eta} = \frac{\mathrm{d}A}{\mathrm{d}\eta} + \pi_s \frac{\mathrm{d}B}{\mathrm{d}\eta}. \tag{54}$$

We have

$$\left(\frac{\partial \xi_3}{\partial s}\right) = \left(\frac{\partial \xi_3}{\partial \pi}\right)\left(\frac{\partial \pi}{\partial \eta}\right) = -\frac{m}{\rho g h_3}, \tag{55}$$

consequently the quantity appearing in (47) is

$$\left(\rho g h_3 \frac{\partial \xi_3}{\partial s}\right) = -m, \tag{56}$$

and the continuity equation writes

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial \xi_1}\left(m\dot{\xi}_1\right)_\eta + \frac{\partial}{\partial \xi_2}\left(m\dot{\xi}_2\right)_\eta + \frac{\partial}{\partial \eta}(m\dot{\eta}) + m\left[\dot{\overline{\left(\frac{h_1 h_2}{g}\right)}}\frac{g}{h_1 h_2}\right] = 0. \tag{57}$$

The vertical derivative of the geopotential is given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi_3} = g h_3, \tag{58}$$

hence

$$\frac{\partial \phi}{\partial \eta} = \frac{\mathrm{d}\phi}{\mathrm{d}\xi_3}\frac{\partial \xi_3}{\partial \eta} = (g h_3)\left(\frac{m}{\rho g h_3}\right) = \frac{m}{\rho}. \tag{59}$$

The vertical integration of the geopotential depth element yields

$$\phi = \phi_s + \int_\eta^1 \frac{m}{\rho}\mathrm{d}\eta = \phi_s + \int_\eta^1 \frac{RT}{p}\mathrm{d}\eta. \tag{60}$$

Here also, in the deep-atmosphere quasi-hydrostatic case, the governing equation system is no easy to close, because $p \neq \pi$.

7 Shallow atmosphere approximation

In the case of the Shallow-Atmosphere (SA) approximation, the metric factors have no vertical dependence. Their value is chosen at some reference level, generally at the mean surface geopotential of the planet, or any other horizontal surface. Hence, we have

$$\left(\frac{\partial h_1}{\partial \xi_3}\right) = \left(\frac{\partial h_2}{\partial \xi_3}\right) = \left(\frac{\partial h_3}{\partial \xi_3}\right) = 0, \tag{61}$$

that is, in other terms

$$h_1 = h_1(\xi_2), \quad h_2 = h_2(\xi_2), \quad h_3 = h_3(\xi_2). \tag{62}$$

Additionally, for dynamical consistency, the gravity is independent of height in the shallow-atmosphere context:

$$\left(\frac{\partial g}{\partial \xi_3}\right) = 0, \quad \text{i.e.} \quad g = g(\xi_2). \tag{63}$$

As a consequence $\mathrm{d}\,(gh_3)/\mathrm{d}\,\xi_3 = 0$, and gh_3 is therefore a true constant

$$\frac{\dot{\overline{gh_3}}}{gh_3} = 0. \tag{64}$$

We also may write

$$\left(\frac{1}{h_3}\frac{\mathrm{d}h_3}{\mathrm{d}\xi_2}\right) = -\left(\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\xi_2}\right). \tag{65}$$

Moreover $(\dot{h}/h)$ reduces to

$$\left(\frac{\dot{h}}{h}\right) = \frac{\dot{\xi}_2}{h}\left(\frac{\mathrm{d}h}{\mathrm{d}\xi_2}\right). \tag{66}$$

Finally, the continuity equation writes

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial\xi_3}{\partial s}\right)_s + \frac{\partial}{\partial\xi_1}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{\xi}_1\right)_s + \frac{\partial}{\partial\xi_2}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{\xi}_2\right)_s + \frac{\partial}{\partial s}\left(\rho\frac{\partial\xi_3}{\partial s}\dot{s}\right) + \rho\frac{\partial\xi_3}{\partial s}\frac{\dot{\xi}_2}{h}\left(\frac{\mathrm{d}h}{\mathrm{d}\xi_2}\right) = 0. \tag{67}$$

which is identical to (38), except that the last rhs term is more simple.

7.1 Terrain-following Hydrostatic-pressure coordinate σ

In Shallow-Atmosphere approximation, when the σ coordinate is used, the continuity equation writes

$$\frac{\partial\pi_s}{\partial t} + \frac{\partial}{\partial\xi_1}\left(\pi_s\dot{\xi}_1\right)_\sigma + \frac{\partial}{\partial\xi_2}\left(\pi_s\dot{\xi}_2\right)_\sigma + \frac{\partial\dot{\sigma}}{\partial\sigma} + \pi_s\frac{\dot{\xi}_2}{h}\left(\frac{\mathrm{d}h}{\mathrm{d}\xi_2}\right) = 0. \tag{68}$$

7.2 Hybrid hydrostatic-pressure terrain-following coordinate η

When $s = \eta$ coordinate is used, the continuity equation writes

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial\xi_1}\left(m\dot{\xi}_1\right)_\eta + \frac{\partial}{\partial\xi_2}\left(m\dot{\xi}_2\right)_\eta + \frac{\partial}{\partial\eta}(m\dot{\eta}) + m\frac{\dot{\xi}_2}{h}\left(\frac{\mathrm{d}h}{\mathrm{d}\xi_2}\right) = 0. \tag{69}$$

8 Shallow atmosphere approximation with “case (ii)" of B14a, B14b

The "case (ii)" of B14a, B14b is a special case of non-spherical geometry which consists in a spheroidal geometry (geopotential surfaces are assumed to be spheroids) but with a spherical planet (the gravity then still may have a varying meridional profile). When this case is combined with the shallow-atmosphere approximation and the reference level for the metric is the planet's surface one, the horizontal metric is the spherical one. One may choose the longitude λ, latitude φ and geopotential ϕ as the (ξ_1, ξ_2, ξ_3) system.
In this case, we have

$$\begin{aligned} h_1 &= a\cos\varphi \\ h_2 &= a \\ h_3 &= h_3(\varphi) = 1/g(\varphi) \\ h &= a^2 h_3(\varphi)\cos\varphi = (a^2\cos\varphi)/g(\varphi), \end{aligned} \tag{70}$$

and we have

$$\frac{\dot{h}}{h} = \frac{\dot{\varphi}}{h}\frac{\mathrm{d}h}{\mathrm{d}\varphi} = -\dot{\varphi}\left(\tan\varphi + \frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right). \tag{71}$$

The physical wind components are

$$u = (a\cos\varphi)\,\dot{\lambda}$$
$$v = a\dot{\varphi}$$

Since the aim of this section is the use of this case in ARPEGE, we only present the η coordinate case below.

8.1 Hybrid hydrostatic-pressure terrain-following coordinate η

In this coordinate, the continuity equation (69) writes

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial\lambda}\left(m\dot{\lambda}\right)_\eta + \frac{\partial}{\partial\varphi}(m\dot{\varphi})_\eta + \frac{\partial}{\partial\eta}(m\dot{\eta}) + m\frac{\dot{\varphi}}{h}\left(\frac{\mathrm{d}h}{\mathrm{d}\varphi}\right) = 0. \tag{72}$$

$\Longrightarrow$

$$\frac{\partial m}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(mu)_\eta + \frac{1}{a}\frac{\partial}{\partial\varphi}(mv)_\eta + \frac{\partial}{\partial\eta}(m\dot{\eta}) - m\frac{v}{a}\left(\tan\varphi + \frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}\varphi}\right) = 0. \tag{73}$$

The way to practically modify ARPEGE to allow this case is examined in a separate memo (file 'EGA_Case2.tex').