AN ALTERNATE WAY TO TREAT HELMHOLTZ EQUATION IN THE NHEE MODEL.

K. YESSAD and F. VOITUS
METEO-FRANCE/CNRM/GMAP/ALGO
We present an alternate way to eliminate equations in the NHEE linear system.

Elimination will be done in order to have horizontal divergence $D'_{t+\Delta t}$ as unknown in the Helmholtz equation, instead of $d_{t+\Delta t}$. 
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Elimination will be done in order to have horizontal divergence $D'_{t+\Delta t}$ as unknown in the Helmholtz equation, instead of $d_{t+\Delta t}$. 
SOME DENOTATIONS.

- $M$ is the mapping factor.
- $\bar{M}$ is a reference mapping factor for semi-implicit computations.
- $a$ is the Earth mean radius.
- $\mathbf{V}$ is the horizontal geographical wind. Components are $U$ and $V$.
- $D'$ is the reduced divergence of horizontal wind.
- $T$ is the temperature.
- $T^*$ is a vertically-constant reference temperature for the semi-implicit scheme.
- $T_a^*$ is a cold vertically-variable reference temperature for the semi-implicit scheme (hidden in linear operators $L^{**}$, $T^{**}$).
- $\Pi$ is the hydrostatic pressure, $\Pi_s$ is the hydrostatic surface pressure.
- $\Pi^*$ is a reference pressure and $\Pi_s^*$ is a reference surface pressure for the semi-implicit scheme.
SOME DENOTATIONS (CONT’D).

- $g$ is the gravity acceleration constant, assumed to be vertically constant in the current documentation.
- $R_d$ the gas constant for dry air.
- $c_{pd}$ is the specific heat at constant pressure for dry air.
- $c_{vd}$ is the specific heat at constant volume for dry air.
- $\kappa_d = R_d / c_{pd}$.
- $\nabla'$ is the reduced first order horizontal gradient.
- $p$ is the total pressure, $p_s$ is the surface total pressure.
- $\hat{Q}$ is the pressure departure variable. Expression of $\hat{Q}$ is $\hat{Q} = \log \frac{p}{\Pi}$.
- $d$ is the vertical divergence.
- $\beta$ : tunable coefficient for the semi-implicit scheme (between 0 and 1).
- $\gamma, \tau, \nu, \mu, G^*, S^*, N^*, L^{**}$ (modified Laplacian operator), $T^{**}$ are generic denotations for linear operators $H, C, N$ are intermediate constants used in the semi-implicit scheme of the NHEE and NHQE models.
DEFINITION OF COR.

- When eliminating equations, quantity COR appears (COR = 0 in the continuous equations).
- Expression of COR is:
  \[
  COR = \frac{c_{vd}}{R_d^2 T^*} \gamma \tau - \frac{c_{vd}}{R_d c_p d} \gamma - \frac{c_{vd}}{R_d T^*} \tau + \frac{c_{vd}}{c_p d} \nu = \frac{c_{vd}}{c_p d} [G^* S^* - S^* - G^* + N^*] \quad (1)
  \]
- COR = 0 when constraint C1 is ensured.
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  \]

- $COR = 0$ when constraint C1 is ensured.
NHEE LINEAR SYSTEM TO BE SOLVED.

\[
\log(\Pi_s)_{t+\Delta t} + \beta \Delta t \nu (\bar{M}^2 D'_t) = \mathcal{P}^* 
\]

\[
D'_t + \beta \Delta t \nabla^2 [\gamma T_{t+\Delta t} - T^*(\gamma \hat{Q}_{t+\Delta t}) + \mu \log(\Pi_s)_{t+\Delta t} + R_d T^* \hat{Q}_{t+\Delta t}] = D'^* 
\]

\[
\hat{Q}_{t+\Delta t} + \beta \Delta t \left[ \frac{c_{pd}}{c_{vd}} (\bar{M}^2 D'_t + d_{t+\Delta t}) - \frac{c_{pd}}{R_d T^*} \tau (\bar{M}^2 D'_t) \right] = \hat{Q}^* 
\]

\[
d_{t+\Delta t} + \beta \Delta t \frac{g^2}{R_d T^*} (L^{**} \hat{Q}_{t+\Delta t}) = \hat{D}^* 
\]

\[
T_{t+\Delta t} + \beta \Delta t \frac{R_d T^*}{c_{vd}} [\bar{M}^2 D'_t + d_{t+\Delta t}] = T^* 
\]
Elimination of $T$, $\hat{Q}$ and $\log(\Pi_s)$ between equations (2), (3), (4) and (6) leads to equation (7):

$$D'_{t+\Delta t} - (\beta \Delta t)^2 \nabla'^2 \left\{ R_d T^* \left( \frac{\gamma}{R_d} - 1 \right) \left( \frac{c_{pd}}{R_d T^*} \tau - \frac{c_{pd}}{c_{vd}} \right) + \frac{R_d T^*}{c_{vd}} \gamma + R_d T^* \nu \right\} \tilde{M}^2 D'_{t+\Delta t} + \{ -R_d T^* \frac{c_{pd}}{c_{vd}} \left( \frac{\gamma}{R_d} - 1 \right) + \frac{R_d T^*}{c_{vd}} \gamma \} d_{t+\Delta t}$$

$$= D'^* + \beta \Delta t \nabla'^2 \left[ R_d T^* \left( \frac{\gamma}{R_d} - 1 \right) \hat{Q}^* - \gamma T^* - R_d T^* P^* \right]$$

(7)

$D'^*$ is defined by equation (8):

$$D'^* = D' + \beta \Delta t \nabla'^2 \left[ R_d T^* \left( \frac{\gamma}{R_d} - 1 \right) \hat{Q}^* - \gamma T^* - R_d T^* P^* \right]$$

(8)

We use the relationship $c_{pd} - c_{vd} = R_d$, the definition of $C$ and we isolate the term $COR$ (see equation (1)) : this equation can be rewritten:

$$\left[ -(\beta \Delta t)^2 \nabla'^2 (C^2 - T^* \gamma) \right] d_{t+\Delta t} + \left[ I - (\beta \Delta t)^2 \nabla'^2 (C^2 (1 + COR)) \tilde{M}^2 \right] D'_{t+\Delta t} = D'^*$$

(9)
Elimination of \( \hat{Q} \) between equations (4) and (5) leads to equation (10):

\[
d_{t+\Delta t} - (\beta \Delta t)^2 \left[ \frac{L^{**}}{H^2} (-c_p \tau + C^2) M^2 D_{t+\Delta t} + \frac{C^2}{H^2} L^{**} d_{t+\Delta t} \right] = \hat{D}^{**} \tag{10}
\]

where \( \hat{D}^{**} \) is defined by:

\[
\hat{D}^{**} = \hat{D}^* + \beta \Delta t \left[ -\frac{g}{H} L^{**} \hat{Q}^* \right] \tag{11}
\]
OLD WAY : HELMHOLTZ EQUATION WITH \(d\) AS UNKNOWN.

When \(\text{COR} = 0\) (constraint C1), elimination of \(D'\) between equations (9) and (10) leads to Helmholtz equation (12):

\[
(I - \beta^2 \Delta t^2 B \bar{M}^2 \nabla'^2) d_{t+\Delta t} = \mathcal{R}
\]

where:

\[
B = C^2 \left( I - \beta^2 \Delta t^2 \frac{L^{**}}{H^2} \right)^{-1} \left( I + \beta^2 \Delta t^2 N^2 T^{**} \right)
\]

and:

\[
\mathcal{R} = \left( I - \beta^2 \Delta t^2 C^2 \frac{L^{**}}{H^2} \right)^{-1} \left[ (I - \beta^2 \Delta t^2 C^2 \bar{M}^2 \nabla'^2) \hat{D}^{**} + \beta^2 \Delta t^2 \frac{L^{**}}{H^2} (-c_p d^* + C^2) \bar{M}^2 D'^{**} \right]
\]
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If we compare with the LHS of the Helmholtz equation in the hydrostatic case:

- The unknown is \( d_{t+\Delta t} \).
- The order of \( \overrightarrow{M}^2 \) and \( \nabla'\nabla' \) is inverted.
- \( B \) now depends on \( \Delta t \), it must be recomputed each time the timestep is changed.

Additional remarks:

- Operator \( T^{**} \) appears, which is simple when constraint C2 is matched (ex: VFD with NDLNPR=1) and more tricky otherwise (ex: VFE-NH).
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- Operator $T^{**}$ appears, which is simple when constraint C2 is matched (ex : VFD with NDLNPR=1) and more tricky otherwise (ex : VFE-NH).
If constraint C1 is not matched (non zero COR).

- This is the case for example for VFE-NH.
- An iterative algorithm has been implemented, with NITERHELM iterations.
- Predictor step: replace COR by 0 and proceed as above.
- Corrector steps: the term containing COR is put in the RHS, it is multiplied by $D'_{t+\Delta t}$ at the previous iteration. The LHS is unchanged, so the elimination and the Helmholtz solving can be done like in the predictor step.
- We rather take as unknowns the increments between the current iteration and the predictor step, that allows to simplify the calculation of the RHS for the corrector step.
If constraint $C_1$ is not matched (non zero $COR$).

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NEW WAY: HELMHOLTZ EQUATION WITH $D'$ AS UNKNOWN.

### General formalism for $D'$ and $d$ equations:

- **Equation (9)** has the following shape:
  \[
  [I + (\beta \Delta t)^2 \nabla'^2 B_1 M^2] D'_{t+\Delta t} + [(\beta \Delta t)^2 \nabla'^2 B_2] d_{t+\Delta t} = D'^{**} \tag{15}
  \]

- **Equation (10)** has the following shape:
  \[
  [(\beta \Delta t)^2 B_3 M^2] D'_{t+\Delta t} + [I + (\beta \Delta t)^2 B_4] d_{t+\Delta t} = \hat{D'^{**}} \tag{16}
  \]

- where:
  
  \[
  \begin{align*}
  B_1 &= -C^2 (1 + COR) \\
  B_2 &= -(C^2 - T^* \gamma) \\
  B_3 &= -(1/H^2) L^{**} (C^2 - c_{pd} \tau) \\
  B_4 &= -(C^2/H^2) L^{**}
  \end{align*}
  \]
NEW WAY (CONT’D).

Elimination:

- The following combination is used:

\[
\text{(EQ 15)} - \left[ (\beta \Delta t)^2 \nabla' B_2 \right] \left[ I + (\beta \Delta t)^2 B_4 \right]^{-1} \text{(EQ 16)}
\]

- And additionally we use the fact that \( \nabla' \) commute with \( B_1, B_2, B_3 \) and \( B_4 \).

- One obtains the following Helmholtz equation:

\[
\left( I - \beta^2 \Delta t^2 B \nabla' \nabla'' \right) D'_t + \Delta t = R \tag{17}
\]

where:

\[
B = -B_1 + (\beta \Delta t)^2 B_2 \left[ I + (\beta \Delta t)^2 B_4 \right]^{-1} B_3 \tag{18}
\]

and:

\[
R = D'''' - (\beta \Delta t)^2 \nabla' B_2 \left[ I + (\beta \Delta t)^2 B_4 \right]^{-1} \hat{D}'''' \tag{19}
\]
Elimination:

- The following combination is used:

\[(\text{EQ 15}) - [((\beta \Delta t)^2 \nabla'^2 B_2)[I + (\beta \Delta t)^2 B_4]^{-1}(\text{EQ 16})]\]

- And additionally we use the fact that \(\nabla'^2\) commute with \(B_1, B_2, B_3\) and \(B_4\).

- One obtains the following Helmholtz equation:

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\[(17)\]

where:

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\[(18)\]

and:

\[
\mathcal{R} = D'^{**} - (\beta \Delta t)^2 \nabla'^2 B_2[I + (\beta \Delta t)^2 B_4]^{-1}\hat{D}^{**}
\]

\[(19)\]
Properties of \( B \): 

- After calculations (not detailed) using expression of \( COR \) and the relationship \( c_{Pd} - c_{vd} = R_d \), \( B \) can be rewritten:

\[
B = B_{\text{hyd}} + B_2\left( (\beta \Delta t)^2[I + (\beta \Delta t)^2 B_4]^{-1} - B_4^{-1}\right) B_3
\]

(20)

where:

\[
B_{\text{hyd}} = \gamma \tau + R_d T^* \nu
\]

- An alternate way to write \( B \) is:

\[
B = B_{\text{hyd}} + \frac{1}{C^2} \left[ C^2 - T^* \gamma \right]\left( (\beta \Delta t)^2 \frac{C^2}{H^2} [I + (\beta \Delta t)^2 B_4]^{-1} L^{**} + I \right][C^2 - c_{Pd} \tau]
\]

(21)

Term \( COR \) and constraint C1 have disappeared.

- For large timesteps, \( B \) converges towards \( B_{\text{hyd}} \).
- For very small timesteps, \( B \) converges towards \( C^2(1 + COR) \).
NEW WAY (CONT’D).

Properties of $B$:

- After calculations (not detailed) using expression of $COR$ and the relationship $c_{pd} - c_{vd} = R_d$, $B$ can be rewritten:

$$B = B_{hyd} + B_2[(\beta \Delta t)^2[I + (\beta \Delta t)^2 B_4]^{-1} - B_4^{-1}]B_3$$

where:

$$B_{hyd} = \gamma \tau + R_d T^* \nu$$

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- For very small timesteps, $B$ converges towards $C^2(1 + COR)$. 
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- For large timesteps, $B$ converges towards $B_{\text{hyd}}$.

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Remarks and advantages:

- Easier to code than initially expected.
- The code is significantly simpler than with the old method.
- Closer to hydrostatic model design.
- Options LSIDG and LIMPF are simpler to implement (easier to re-use hydrostatic model pieces of code).
- No C1 constraint, quantity COR does not appear any longer.
- No iterative algorithm for VFE-NH.
- Operator $T^{**}$ does not appear any longer.
- $B$ easily writes as a sum of hydrostatic and anhydrostatic contributions.
- Calculations use quantity $[I + (\beta \Delta t)^2 B_4]^{-1}$ which is already computed with the old method (array SIFACI).
- It is actually surprising that this solution has not been thought and coded before year 2007 !!!
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REPRODUCIBILITY BETWEEN NEW WAY RESULTS AND OLD WAY RESULTS:

Some results:

- Constraint C1 matched (VFD): small numerical differences.
- Constraint C1 not matched (VFE-NH): the old algorithm with \( \text{NITERHELM} > 10 \) converges towards the solution given by the new one.
Figure – OC0500 : HU850 range 6h ; REF (init 31/03/2015 00TU).
Figure – OC0500 : HU850 range 6h ; SINHEE (init 31/03/2015 00TU).
NEW WAY : CALENDAR OF IMPLEMENTATION.

Calendar:
- It is coded on the top of CY45T1 for both global and LAM models.
- Expected to enter the code in CY46T1 and CY47.