Exploring some alternatives to improve the robustness of mass-based SI Spectral NH system

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Motivations and Objectives

Motivations

Allow very high-resolution (steep orography) and attractive time-steps for NWP, but still in the framework of the current constant-coefficient Semi-implicit approach.

Explored Avenues

1. Design of a modern Sound-proof NH approximate set of equations less stiff than fully-compressible (EE) system by exploiting Arakawa and Konor (2009) ⇒ Suppression of high-frequency vertically-propagating acoustic wave at their source ⇒ Potential benefit in terms of stability.

2. Design of a new prognostic variable for the EE system along the lines of the $d_4$ variable of Bénard et al. (2005), leading to a more stable constant-coefficient semi-implicit time scheme over steep slopes.
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Plan

1. SWITCH TO (MODERN) SOUND-PROOF NH EQUATIONS

2. A MORE ROBUST VERTICAL MOMENTUM PROGNOSTIC VARIABLE
Switch to modern Sound-proof NH equations

Definition of pseudo-hydrostatic QE reference-state: \((\pi, \tilde{\rho}, \tilde{T})\)

\[
\begin{align*}
\tilde{\rho} &= \frac{\pi}{RT} \left(\frac{p}{\pi}\right)^{R/C_p}, \\
\frac{\partial\pi}{\partial z} &= -g\tilde{\rho} \\
\tilde{T} &= \frac{\pi}{\tilde{\rho}R}
\end{align*}
\]

Determination of actual thermodynamic state: \((p, \rho, T)\)

Defining the pressure departure as \(\hat{q} = \log(p/\pi)\), it yields

\[
\begin{align*}
p &= \pi \exp[\hat{q}], \\
\rho &= \tilde{\rho} \exp\left[\left(\frac{C_v}{C_p}\right)\hat{q}\right], \\
T &= \tilde{T} \exp[\left(\frac{R}{C_p}\right)\hat{q}].
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Switch to modern Sound-proof NH equations

**QE approximation : Basic underlying idea**

- Mass-continuity Eq. for $\tilde{\rho}$:

\[
\frac{D\tilde{\rho}}{Dt} = \left( \frac{\partial \rho}{\partial \Pi} \right)_{\theta, \hat{q}} \frac{D\Pi}{Dt} + \left( \frac{\partial \rho}{\partial \hat{q}} \right)_{\theta, \Pi} \frac{D\hat{q}}{Dt} = -\tilde{\rho} D_3
\]

- NH compressibility of the fluid is neglected in mass continuity Eq.
  \( \Rightarrow \) Minimal condition for filtering vertically propagating acoustic waves,

- Hydrostatic compressibility is maintained for good accuracy at large-scales.

\[
\left( \frac{\partial \tilde{\rho}}{\partial \hat{q}} \right)_{\theta, \Pi} \frac{D\hat{q}}{Dt} = D_3 + \frac{C_v}{C_p} \left( \hat{\Pi} \right) = 0
\]

- $\hat{q}$ becomes a diagnostic variable.
Switch to modern Sound-proof NH equations

QE approximation : Basic underlying idea

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\textbf{H—compressibility} \hspace{1cm} \textbf{NH—compressibility}

1. NH compressibility of the fluid is neglected in mass continuity Eq.
   ⇒ Minimal condition for filtering vertically propagating acoustic waves,

2. Hydrostatic compressibility is maintained for good accuracy at large-scales.

$$\left(\frac{\partial\tilde{\rho}}{\partial\hat{q}}\right)_{\theta,\pi} \frac{D\hat{q}}{Dt} = D_3 + \frac{C_v}{C_p} \left(\frac{\dot{\pi}}{\pi}\right) = 0$$

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Switch to modern Sound-proof NH equations

QE divergence constraint:

\[
D - \frac{C_v}{C_p} \frac{1}{\pi} \int_{0}^{\pi} D \, d\pi' + d_4 = 0
\]
Switch to modern Sound-proof NH equations

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\[
D - \frac{C_v}{C_p} \frac{1}{\pi} \int_0^\pi D d\pi' + d_4 = 0
\]

where \( D = \nabla \cdot V \), and

\[
d_4 = -\frac{g}{\tilde{R} \tilde{T} \tilde{m}} \frac{\pi}{\partial \eta} \partial w + \left\{ \frac{\pi}{\tilde{R} \tilde{T} \tilde{m}} \frac{\partial V}{\partial \eta} \cdot \nabla \Phi + \frac{C_v}{C_p} \left[ \frac{V \cdot \nabla \pi}{\pi} - \frac{1}{\pi} \int_0^{\eta} (V \cdot \nabla m) \, d\eta' \right] \right\}
\]

\( d_3 \)
Switch to modern Sound-proof NH equations

QE divergence constraint:

\[
D - \frac{C_v}{C_p} \frac{1}{\pi} \int_0^\pi D \, d\pi' + d_4 = 0
\]

where \( D = \nabla \cdot V \), and

\[
d_4 = -\frac{g}{R \tilde{T}} \frac{\pi}{m} \frac{\partial w}{\partial \eta} + \left\{ X_S \right\} + \left\{ \frac{\pi}{R \tilde{T} m} \frac{\partial V}{\partial \eta} \cdot \nabla \Phi + \frac{C_v}{C_p} \left[ V \cdot \frac{\nabla \pi}{\pi} - \frac{1}{\pi} \int_0^\eta (V \cdot \nabla m) \, d\eta' \right] \right\}
\]

\[
\frac{D \, d_4}{Dt} = -\frac{g}{R \tilde{T}} \frac{\pi}{m} \left[ \frac{\partial}{\partial \eta} \left( \frac{dw}{dt} \right) - \frac{\partial V}{\partial \eta} \cdot \nabla w \right]
\]

\[
+ (X - d_4) (d_4 - X + X_S) + \dot{X}
\]
Symbolical description of QE adiabatic system

- Let us denote the state-vector $\mathbf{z} = (\mathbf{x}, \hat{q})$.
  
  $\mathbf{x} = (V, d_4, \tilde{T}, \log \pi_s)$ is the vector of prognostic variables.

- Prognostic Eqs.:

  \[
  \frac{\partial \mathbf{x}}{\partial t} = \mathcal{A} (\mathbf{x}) + \mathcal{M} (\mathbf{z}) \quad \text{(Eulerian form)}
  \]

  \[
  \frac{D \mathbf{x}}{D t} = \mathcal{M} (\mathbf{z}), \quad \text{(Lagrangian form)}
  \]

- QE constraint:

  \[
  D (\mathbf{x}) = 0
  \]

- $(\mathcal{L}^*, \mathcal{C}^*)$ respectively denote the linear counterpart operators of $(\mathcal{M}, D)$ around a constant-coefficient SI-background $\mathbf{z}^*$. 
Switch to modern Sound-proof NH equations

3-TL SI Eulerian time discretization

For $\nu \in [0, N_{iterhelm} - 1]$ (inner-loop)

- Time-discrete prognostic Eqs. :
  \[
  \frac{X^{+(\nu)} - X^-}{2\Delta t} = A(X^0) + M(Z^0) - L^*.Z^0
  + \frac{L^*.Z^{+(\nu)} + L^*.Z^-}{2}
  \]

- Newton-like iterative treatment of QE constraint :
  \[
  C^*X^{+(\nu)} = C^*X^{+(\nu-1)} - D[X^{+(\nu-1)}]
  \]
Switch to modern Sound-proof NH equations

Extension to 2-TL ICI SL time-discretization

For $i \in [1, N_{\text{iter}}]$ (outer-loop)

For $\nu \in [0, N_{\text{iterhelm}} - 1]$ (inner-loop)

• Time-discrete prognostic Eqs. :

$$\frac{X_F^{+(i,\nu)} - X_{O(i-1)}}{\Delta t} = \frac{M \left[ Z^{+(i-1)} \right]_F + M \left[ Z^0 \right]_{O(i-1)}}{2} + \frac{L^* \cdot Z_F^{+(i,\nu)} - L^* \cdot Z_F^{+(i-1)}}{2}$$

• QE constraint iterative treatment:

$$C^* \cdot X_F^{+(i,\nu)} = C^* \cdot X_F^{+(i,\nu-1)} - D \left[ X^{+(i,\nu-1)} \right]_F$$
Switch to modern Sound-proof NH equations

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$$+ \frac{L^* \cdot Z_F^{+(i,\nu)} - L^* \cdot Z_F^{+(i-1)}}{2}$$

• QE constraint iterative treatment:

$$C^* \cdot X_F^{+(i,\nu)} = C^* \cdot X_F^{+(i,\nu-1)} - D \left[ X^{+(i,\nu-1)} \right]_F + \left[ D \left( X^0 \right) \right]_{O(i-1)}$$
Linear stability analysis with orography

- Time-discrete space-continuous linear stability analysis of 3-TL SI scheme with a uniform sloped orography (without advection). Settings: $\Delta x = 2000$ m, $\Delta t = 200$ s.

- Amplification factor as function of the slope, and the non-linear thermal residual factor:

$$\alpha = \frac{(T - T^*)}{T^*}$$
Idealized 2D test-cases: 3-TL SI Eulerian scheme

- **Potential flow 2D test-case**: Basic-state is defined by: $U = 15$ m/s, $N = 0.02$ s$^{-1}$, maximum height and half-width of the agnesi-mountain are both equal to 100 m ⇒ maximum slope of $33^\circ$. Settings: $\Delta x = 10$ m, and $\Delta \eta$ is chosen in such a way that $\Delta z \approx 15$ m, $\Delta t = 0.25$ s, SITR=$350$ K, SITRA = 35 K, and $N_{\text{iterhelm}} = 1$. 

![Graph](image1.png)

![Graph](image2.png)
Coding and Validations

**Status**

- QE code in available in cycle 46 (95% of QE code).
- Validation using "mitraillette-test" are now in progress.

- Obviously, there is still some bugs to deal with !!!
Plan

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2. A MORE ROBUST VERTICAL MOMENTUM PROGNOSTIC VARIABLE
A potential cause of poor stability over steep slopes

Asymmetric rigid BBC time treatment

- Non-homogeneous rigid BBC specification ⇒ a distinct BBC treatment between the explicit grid-point part and the implicit spectral part.

\[ w_S = \frac{1}{g} (V_S \cdot \nabla \Phi_S), \]

\[ \dot{w}_S = \frac{1}{g} \left[ \dot{V}_S \cdot \nabla \Phi_S + V_S \cdot \nabla (V_S \cdot \nabla \Phi_S) \right] \]

- Need for an extra Free-slip BBC: \( V_S = V_{NLEV} \)

Rigid BBC in NL model \( \mathcal{M} \):

Rigid BBC linear model \( \mathcal{L}^* \):

\[ w_S = 0 \]

\[ \dot{w}_S = 0 \]
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Rigid BBC in NL model $\mathcal{M}$:

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\omega_S &= \frac{1}{g} (V_S \cdot \nabla \Phi_S), \\
\dot{\omega}_S &= \frac{1}{g} \left[ \dot{V}_S \cdot \nabla \Phi_S + V_S \cdot \nabla (V_S \cdot \nabla \Phi_S) \right]
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\end{align*}
\]

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\end{align*}
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\begin{align*}
\omega_S &= 0 \\
\dot{\omega}_S &= 0
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\]

- Need for an extra Free-slip BBC: $V_S = V_{\text{NLEV}}$
Imposing homogeneous rigid BBC

Let us consider the following variable:

\[
W = \frac{(\partial_\eta \Phi)}{g} \left[ \dot{\eta} + \frac{(\partial_t \Phi)}{(\partial_\eta \Phi)} \right] = w - \frac{1}{g} (V \cdot \nabla \Phi)
\]

⇒ Rigid BBC becomes:

\[
W_S = 0
\]
\[
\dot{W}_S = 0
\]

⇒ At the top, material condition leads to

\[
W_T = \frac{\partial_t \Phi_T}{g}
\]
A more robust prognostic variable over steep slopes

Use of a new vertical divergence variable

Let us define

\[ d_5 = -\frac{g \rho}{m} \left( \partial_\eta \mathbb{W} \right) - \frac{\rho}{m} V \cdot \nabla \left( \partial_\eta \Phi \right) \]

\[ \Rightarrow \quad D_3 = D + d_5 \]

- Prognostic Eq. for \( d_5 \):

\[ \frac{D d_5}{Dt} = -\frac{g \rho}{m} \left[ \frac{\partial}{\partial \eta} \left( \frac{\partial \mathbb{W}}{\partial t} \right) - \frac{\partial V}{\partial \eta} \cdot \nabla \mathbb{W} \right] \]

\[ + (X - d_5) d_5 + \dot{X} \]
A more robust prognostic variable over steep slopes

- Governing equation of $\mathbb{W}$:

$$\frac{D\mathbb{W}}{Dt} = \frac{Dw}{Dt} - \frac{1}{g} \frac{D[V \cdot \nabla \Phi]}{Dt}$$

**Eulerian explicit approach**

$$\frac{D\mathbb{W}}{Dt} = \mathcal{M}_w(x) - \left[ \dot{V} - (V \cdot \nabla) V - \dot{\eta} (\partial_\eta V) \right] \cdot \frac{\nabla \Phi}{g}$$

$$- V \cdot \nabla \left[ w - \frac{\dot{\eta} (\partial_\eta \Phi)}{g} \right] - \dot{\eta} \partial_\eta \left[ V \cdot \frac{\nabla \Phi}{g} \right]$$

* Require extra spectral transforms to compute $\nabla[V \cdot \nabla \Phi]$ and $\nabla[\dot{\eta} (\partial_\eta \Phi)]$
A more robust prognostic variable over steep slopes

- Governing equation of $W$

\[
\frac{DW}{Dt} = \frac{Dw}{Dt} - \frac{1}{g} \frac{D[V \cdot \nabla \Phi]}{Dt}
\]

ICI Semi-Lagrangian approach (in the spirit of LGWADV option)

\[
\frac{DW}{Dt} = \frac{1}{2} \left\{ M_w (x) - \frac{2[V \cdot \nabla \Phi]}{g \Delta t} \right\}^+_F + (i-1)
\]
\[
+ \frac{1}{2} \left\{ M_w (x) + \frac{2[V \cdot \nabla \Phi]}{g \Delta t} \right\}^0_{O(i-1)}
\]

☆ May require some extra SL interpolations for $[V \cdot \nabla \Phi]$. 
A more robust prognostic variable over step slopes

- Fully-discrete linear stability analysis of 2-TL ICI ($N_{siter} = 1$) with a prescribed sinusoidal orography (without advection).
- Amplification factor as function of horizontal wave Courant number for three different slopes: 15°, 25°, and 45°. For current $d_4$ (left panel)
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A more robust prognostic variable over step slopes

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- Amplification factor as function of horizontal wave Courant number for three different slopes: 15°, 25°, and 45°. For current $d_4$ (left panel), with new $d_5$ (right panel).
Idealized 2D test-cases: 3-TL SI Eulerian scheme with $d_5$

- Potential flow 2D test-case: re-run for EE system with $d_5$
Summary and perspectives

Summary

- Can we still improve the stability of the constant-coefficient SI spectral NH system? ⇒ Yes, we can!
- Switch to QE system may provide a substantial gain in stability.
- New variable $d_5$ is very promising for EE system, and can also be extended to QE system.
- There is always a price to pay, nothing is given for free: extra spectral transforms, extra SL interpolations.

Perspectives

- Validation of QE code will be pursued.
- Coding of this new variable $d_5$ should be envisaged.
Thanks for your attention !!!